The moduli stack $\mathcal{M}_g$

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1 Introduction

Many interesting types of objects in algebraic geometry are parametrized in a natural way by geometric objects, called "moduli schemes" or "moduli spaces". Here we are interested in the case of smooth, proper curves of genus g. For these we have a notion of "coarse moduli scheme", usually denoted M_g, which does not have the good properties that a "fine moduli scheme" would have. For example, to a k-point of M_g one cannot always associate a curve defined over k, but rather a curve defined over k^{sep} and isomorphic to its Galois conjugates. Hence, $M_0(\mathbb{R})$ consists of only one point although there exists more than one isomorphism class of curves of genus zero defined over $\mathbb{R}$. For instance, the conic $V(x^2+y^2+z^2) \subset \mathbb{P}^2_{\mathbb{R}}$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$.

On the other hand, by definition, the k-points of a fine moduli scheme correspond bijectively and functorially to (isomorphism classes of) objects defined over k. However, it turns out there cannot exist a fine moduli scheme for smooth curves. This happens more in general when the objects we consider have non-trivial automorphisms, as in the case of curves. Nonetheless, if we enlarge the category where we would like our moduli space to live (to the point where it is not a category anymore, but a 2-category), and introduce stacks, we can still find objects enjoying all the properties that we expect from a fine moduli space. The downside of stacks is that geometric intuition often breaks down when working with them (for example, they can have negative dimension). In fact, they are not even topological spaces. Among their advantages, we count the fact that they provide an optimal setting where to work with moduli problems, and that in the 2-category of stacks there is a good notion of quotient by an action of a smooth algebraic group.

2 Non-existence of a fine moduli scheme

We start by giving a precise definition of the objects in which we are interested.

Definition 2.1. Let S be a scheme and g a non-negative integer. We call a morphism of schemes $C \to S$ a *curve* if it is proper, smooth, and all geometric fibres are connected schemes of dimension 1. We say that $C \to S$ has genus g if all the geometric fibres have genus g.

If the scheme S is the spectrum of a field, we recover the usual notion of curve over a field. When S is a more general scheme, we can interpret $C \to S$ as a family of curves parametrized by S. We point out that the smoothness condition implies that $C \to S$ is flat morphism. This ensures that the fibres of the family vary well (for example, their genus is automatically locally constant on S). In fact, the flatness condition is usually imposed in all kinds of moduli problems when defining families of objects.

We investigate whether a "fine moduli scheme" for smooth curves of genus g can exist. Consider the functor

$$\begin{aligned} F_g : \underline{\mathrm{Sch}}^{opp} &\to \mathbf{Sets} \\ S &\mapsto \{\text{Curves } C \to S \text{ of genus } g\}/\cong \\ T \to S &\mapsto [C \to S] \mapsto [C \times_S T \to T] \end{aligned}$$

Definition 2.2. A *fine moduli scheme* for F_g is a couple $(\mathcal{M}_g, \varphi)$ of a scheme $\mathcal{M}_g$ and an isomorphism of functors

$$\varphi : \mathrm{Hom}(\_\_, \mathcal{M}_g) \to F_g$$

A nice consequence of the existence of a fine moduli scheme would be that we would obtain a distinguished curve $\mathcal{C} \to \mathcal{M}_g$ in $F_g(\mathcal{M}_g)$ corresponding via φ to the identity $\mathcal{M}_g \to \mathcal{M}_g$, with the property that every curve $C \to S$ arises as pullback of $\mathcal{C} \to \mathcal{M}_g$ via the corresponding map $S \to \mathcal{M}_g$. One calls the curve $\mathcal{C} \to \mathcal{M}_g$, the *universal curve of genus g*.

However, the functor F_g is not representable. In fact, it is not even a Zariski-sheaf, hence a scheme $\mathcal{M}_g$ satisfying Definition 2.2 cannot exist. In the following examples we show why.

Example 2.3. Let E be (the scheme underlying) an elliptic curve over a field k, and let $E \xrightarrow{\sigma} E$ be a non-trivial automorphism. Consider two copies $Y \to \mathbb{P}^1_k$ and $Z \to \mathbb{P}^1_k$ of the constant family $E \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$. Now glue the two base $\mathbb{P}^1_k$'s at the point 0 and at the point ∞, and call S the resulting scheme. Next glue the fibres $Y_0 = E$ and $Z_0 = E$ via the identity, and the fibres $Y_\infty = E$ and $Z_\infty = E$ via the automorphism σ. We obtain a curve $f : X \to S$,. Compare it to the constant family $g : E \times_k S \to S$. The two families are not isomorphic, but they become isomorphic when restricted to both of the covering open subschemes $S \setminus \{0\}$ and $S \setminus \{\infty\}$ of S. So we see that F_g is not a Zariski sheaf.

Example 2.4. The set $F_0(\operatorname{Spec} \mathbb{C})$ has only one element, corresponding to the isomorphism class of $\mathbb{P}^1_{\mathbb{C}}$. The set $F_0(\operatorname{Spec} \mathbb{R})$ has more than one element, since there exist genus zero curves with no $\mathbb{R}$-points (such as the conic $V(x^2+y^2+z^2) \subset \mathbb{P}^2_{\mathbb{R}}$). however, if F_0 were representable by $\mathcal{M}_0$, there would have to be an injection $\mathcal{M}_0(\mathbb{R}) \hookrightarrow \mathcal{M}_0(\mathbb{C})$.

The examples show that the moduli functor F_g is not representable. As suggested by Example 2.3, the obstruction to its representability lies in the existence of non-trivial automorphisms of curves, which are forgotten by the functor F_g. To remedy to this problem, we need to keep track of isomorphisms of curves, so the first thing to do is to try and redefine the functor F_g so that it takes values in categories rather than in sets.

3 The category fibered in groupoids $\mathcal{M}_g \to \underline{\mathrm{Sch}}$

Definition 3.1. Fix a non-negative integer g. We let $\mathcal{M}_{\mathrm{g}}$ be the following category:

- its objects are couples $(S, C \to S)$ with S a scheme and $C \to S$ a curve of genus g.
- its morphisms $(S, C \to S) \to (T, D \to T)$ correspond to cartesian diagrams

$$\begin{array}{ccc} D & \longrightarrow & T \\ \downarrow & & \downarrow \\ C & \longrightarrow & S \end{array}$$

The category M_g is naturally endowed with a forgetful functor $p : \mathcal{M}_{\mathrm{g}} \to \underline{\mathrm{Sch}}$, associating to an object $(S, C \to S)$ the base scheme S over which the curve C is defined, and to an arrow the right vertical map in the cartesian diagram above. We can now start to make sense of how to interpret $\mathcal{M}_{\mathrm{g}}$ as a "functor of categories":

Definition 3.2. Let S be an object of $\underline{\mathrm{Sch}}$. The category $\mathcal{M}_{\mathrm{g}}(S)$ is the subcategory of $\mathcal{M}_g$ defined in the following way:

- its objects are the objects x of $\mathcal{M}_{\mathrm{g}}$ such that $p(x) = S$, namely those of the form $(S, C \to S)$.
- its morphisms are the morphisms f in $\mathcal{M}_{\mathrm{g}}$ such that $p(f) = \mathrm{id}_S$

We call $\mathcal{M}_{\mathrm{g}}(S)$ the *fiber category* of $\mathcal{M}_{\mathrm{g}}$ over S.

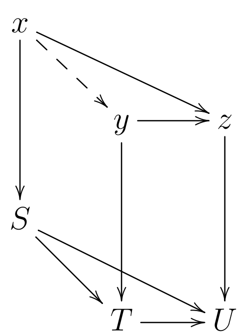
A morphism $(S, C) \to (S, C')$ in $\mathcal{M}_{\mathrm{g}}(S)$ is nothing else but the datum of an isomorphism of S-schemes $C' \to C$. Therefore all arrows in the fiber category $\mathcal{M}_{\mathrm{g}}(S)$ are isomorphisms. A category with such a property is called a *groupoid.* We say that $\mathcal{M}_{\mathrm{g}} \to \underline{\mathrm{Sch}}$ is a *category fibered in groupoids.*

Definition 3.3. A *category fibered in groupoids* over $\underline{\mathrm{Sch}}$ is a category $\mathcal{X}$ together with a functor $p : \mathcal{X} \to \underline{\mathrm{Sch}}$ satisfying the following properties:

1. For every morphism of schemes $T \to S$ and object x of $\mathcal{X}$ over S, there exists an object y of $\mathcal{X}$ completing the diagram

$$\begin{array}{ccc} y & \dashrightarrow & x \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

2. For all diagrams



there exists a unique arrow $x \to y$ over $S \to T$ filling in the diagram.

For all schemes S we write $\mathcal{X}(S)$ for the fiber category of objects of $\mathcal{X}$ over S, with morphisms lying over the identity of S.

Remark 3.4. Property 1) is about existence of pullbacks, whereas property 2) implies that pullbacks are unique up to unique isomorphism, and also that the fiber categories $\mathcal{X}(S)$ are groupoids.

To every functor $F : \underline{\mathrm{Sch}}^{opp} \to \mathbf{Sets}$ we can associate a category fibered in groupoids $\mathcal{X}_F \to \underline{\mathrm{Sch}}$. Indeed, to a set one can always associate a category with objects the elements of the set and with only the identities as arrows. So for every scheme T we ask that the fiber category $\mathcal{X}_F(T)$ be the category associated to the set $F(T)$. We give a more precise definition.

Definition 3.5. Let $F : \underline{\mathrm{Sch}}^{opp} \to \mathbf{Sets}$. We let $\mathcal{X}_F$ be the category with:

- as objects, pairs (S, x) with S a scheme and $x \in F(S)$;
- as arrows $(S, x) \to (T, y)$, morphisms of schemes $f : S \to T$ such that $F(f)(y) = x$.

As a particular case of this construction, one can see any scheme S as a functor via the Yoneda embedding and hence as a category fibered in groupoids $\mathcal{X}_S$ over $\underline{\mathrm{Sch}}$. Its objects are couples $(T, T \to S)$ of a scheme T and a morphism $T \to S$, its arrows $(T, T \to S) \to (U, U \to S)$ are S-morphisms $T \to U$.

Categories fibered in groupoids over $\underline{\mathrm{Sch}}$ form a 2-category, which we call $\underline{\mathrm{CFG}}$. Objects are categories fibered in groupoids over $\underline{\mathrm{Sch}}$, 1-arrows $(p : \mathcal{X} \to \underline{\mathrm{Sch}}) \to (q : \mathcal{Y} \to \underline{\mathrm{Sch}})$ are functors $d : \mathcal{X} \to \mathcal{Y}$ such that the equality of functors $q \circ d = p$ holds, and 2-arrows are natural transformations of 1-arrows.

We mention the following 2-categorical version of Yoneda's lemma: for every scheme S and category fibered in groupoids $\mathcal{Y} \to \underline{\mathrm{Sch}}$ there is an equivalence of categories

$$\mathrm{Hom}_{\underline{\mathrm{CFG}}}(\mathcal{X}_S, \mathcal{Y}) \cong \mathcal{Y}(S).$$

Given a functor $d : \mathcal{X}_S \to \mathcal{Y}$, we associate to it the object $d(S, S \xrightarrow{\mathrm{id}} S)$ of $\mathcal{Y}(S)$. When plugging in $\mathcal{M}_g$ in place of $\mathcal{Y}$, we recover the correspondence between curves over S and maps from a scheme S to $\mathcal{M}_g$, this time under the form of an equivalence of categories rather than a bijection of sets.

4 $\mathcal{M}_g$ is a stack if $g \neq 1$

So far we have mostly done "abstract nonsense". The step from categories fibered in groupoids to stacks is where geometry finally comes in, and may be interpreted as the step from "functors, or presheaves, with values in categories", to "sheaves with values in categories". Indeed, that $\mathcal{M}_g$ is a stack means that one can reconstruct morphisms and objects of $\mathcal{M}_g$ from compatible local data, just as for a sheaf it is possible to reconstruct global sections from compatible local sections. To make sense of this sort of statements, we have to fix a Grothendieck topology on $\underline{\mathrm{Sch}}$. We choose the fppf-topology.

So let's go back to the category fibred in groupoids $\mathcal{M}_{\mathrm{g}} \to \underline{\mathrm{Sch}}$. Proposition 4.1 and Theorem 4.3 prove that $\mathcal{M}_g$ is an fppf-stack for $g \neq 1$.

Proposition 4.1. *Let S be a scheme and write (abusively) C, C' to denote two objects of $\mathcal{M}_{\mathrm{g}}(S)$. Consider the functor* [1]

$$\begin{array}{rcl} \underline{\mathrm{Isom}}_S(C, C') : (\underline{\mathrm{Sch}}\,/S)^{opp} & \to & \mathbf{Sets} \\ T & \mapsto & \mathrm{Isom}_T(C_T, C'_T) \end{array}$$

Then $\underline{\mathrm{Isom}}_{\mathcal{M}_g}(C, C')$ is a sheaf for the fppf topology.

Proof. Every S-scheme is a sheaf for the fppf topology on $\underline{\mathrm{Sch}}\,/S$ (Theorem 2.55, [FGA]), hence the functor $T \mapsto \mathrm{Hom}_S(T, C')$ is a sheaf. It follows that the functor $\underline{\mathrm{Hom}}_S(C, C')$ is a sheaf, using that for every fppf-cover $\{U_i \to T\}$ the pullback $\{C_{U_i} \to C_T\}$ is also an fppf-cover. To conclude we observe that the property of being an isomorphism is fppf-local. $\square$

This simply means that given two curves C, C' over a scheme S, an fppf-cover $\{S_i \to S\}_{i \in I}$ and S_i-maps $\varphi_i : C \times_S S_i \to C' \times_S S_i$ that agree on the intersections $S_i \times_S S_j$, we are able to recover a unique global map $\varphi : C \to C'$ with $\varphi_{|S_i} = \varphi_i$ for all $i \in I$.

In order to glue objects, the situation is somewhat more involved: suppose we have a cover of a scheme $\{S_i \to S\}$, and objects $C_i \to S_i$. In order to glue them to a global object $C \to S$, we need, first of all, isomorphisms on the intersections $\varphi_{ij} : (C_i)_{|S_j} \to (C_j)_{|S_i}$. Moreover these isomorphisms need to be compatible on triple intersections. It turns out that from these data one can recover a curve $C \to S$ only in the case of (non-negative) genus unequal to 1.

Definition 4.2. Let $\{S_i \to S\}_{i \in I}$ be an fppf-cover. For each $i \in I$ let $f_i : C_i \to S_i$ be an object of $\mathcal{M}_{\mathrm{g}}(S_i)$, and let

$$\varphi_{ij} : (C_i)_{|S_i \times_S S_j} \to (C_j)_{|S_i \times_S S_j}$$

be isomorphisms of $S_i \times_S S_j$-schemes, for $i, j \in I$. Suppose that for all i, j, k, we have on $S_i \times_S S_j \times_S S_k$

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}.$$

Then the set $\{(f_i, \varphi_{i,j}\}_{i,j \in I}$ is called a *fppf-descent datum* for $\mathcal{M}_g$ relative to the cover $\{S_i \to S\}$. We say that $\{(f_i, \varphi_{i,j})\}_{i,j \in I}$ is *effective* if there exists a curve $f : C \to S$ with isomorphisms $\alpha_i : C_i \to C_{|S_i}$ for all $i \in I$ such that for all $i, j \in I$ the diagrams

$$\begin{array}{ccc} C_{i|S_i \times S_j} & \xrightarrow{\varphi_{ij}} & C_{j|S_i \times S_j} \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_j} \\ C_{|S_i \times S_j} & \xrightarrow{\mathrm{id}} & C_{|S_i \times S_j} \end{array}$$

commute.

Theorem 4.3. *If $g \neq 1$, all fppf-descent data for $\mathcal{M}_g$ are effective.*

To prove Theorem 4.3, we will make heavy use of effectiveness of fppf-descent for quasi-coherent sheaves, which we will not prove.

[1]that this is a functor follows from the fact that for any map $U \to T$ of S-schemes, $(C \times_S T) \times_T U$ and $C \times_S U$ are naturally isomorphic.

Fact 4.4 ([S],TAG 023T or Theorem 2.2 [Z]). *The category fibered in groupoids* QCoh $\to$ Sch, *whose objects are pairs* $(T, \mathcal{F})$ *of a scheme* T *and a quasi-coherent sheaf* $\mathcal{F}$ *on it, satisfies the analogous of Proposition 4.1 and Theorem 4.3. In other words, every fppf-descent datum of quasi-coherent sheaves is effective, and for every two quasi-coherent sheaves* $\mathcal{F}$ *and* $\mathcal{G}$ *on* T*, the functor* $\mathrm{Isom}_T(\mathcal{F}, \mathcal{G})$ *is a fppf-sheaf.*

Proof of Theorem 4.3. If the cover $\{S_i \to S\}$ is a Zariski cover, then the statement is the content of Exercise 2.12 pag, 80 [H]. Given this, we show that we can reduce to checking effectiveness of descent for covers given by one faithfully flat map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ of finite presentation. The argument is the following: if $\{S_i \to S\}$ is an fppf-cover, so is $T = \bigcup_i S_i \to S$; the scheme S has a Zariski cover by affine opens, so we reduce to the case of S affine. We can cover T with affine opens $\{V_i\}_{i\in I}$, whose images U_i in S are open because the map $T \to S$ is faithfully flat of finite presentation, hence open. Then a finite subcover $\{U_i\}_{i\in F}$, $F \subseteq I$ of the U_i covers S, by quasi-compactness of S. Hence the finite set $\{V_i\}_{i\in F}$ covers T. The disjoint union $S' := \sqcup_{i\in F} V_i$ is affine, and the map $S' \to S$ is a faithfully flat cover, so we have reduced to the case that we wanted. Moreover the descent datum on $\{S_i \to S\}$ induces a descent datum on $S' \to S$.

Now let $f : C' \to S'$ be a genus $g \neq 1$ curve, and $\varphi : p_1^* C' \to p_2^* C'$ an isomorphism satisfying the cocycle condition, where p_1 and p_2 are the two maps $S' \times_S S' \to S'$. Define the locally-free sheaf $\mathcal{F}$ on C' to be:

- ${\Omega^1_{C'/S'}}^{\otimes -1}$ if $g = 0$;
- ${\Omega^1_{C'/S'}}^{\otimes 3}$ if $g \geq 2$.

Then $\mathcal{F}'$ is very ample relatively to S' (see [Z], Theorem 2.9.), and we set $\mathcal{E}' := f_*(\mathcal{F}')$. The very ample sheaf $\mathcal{F}'$ defines a closed immersion $C' \to \mathbb{P}(\mathcal{E}')$, given by an homogeneous sheaf of ideals $\mathcal{I}'$ in $\mathbb{P}(\mathcal{E}')$. Since pushforward commutes with flat base-change, from the descent datum φ we obtain a descent datum of quasi-coherent sheaves $\mathcal{E}'$ relative to $S' \to S$. By Fact 4.4 we obtain a quasi-coherent sheaf $\mathcal{E}$ on S. The morphism $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$ is faithfully flat (because taking the Proj commutes with base change), hence we obtain a descent datum for the sheaf $\mathcal{I}'$ relative to the fppf-cover $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$ and therefore a sheaf $\mathcal{I}$ on $\mathbb{P}(\mathcal{E})$. The injective morphism $\mathcal{I}' \to \mathcal{O}_{\mathbb{P}(\mathcal{E}')}$ descends to a morphism $\mathcal{I} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$. Because $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$ is faithfully flat, $\mathcal{I} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ is also injective. It is easily verified that the closed subscheme $C \subset \mathbb{P}(\mathcal{E})$ defined by $\mathcal{I}$ is a curve of genus g over S. □

Any category fibered in groupoids satisfying the analogous of Prop. 4.1 and Theorem 4.3 is called an *fppf-stack* over Sch. So we have just seen that $\mathcal{M}_g$ is an fppf-stack for $g \neq 1$. By fact 4.4, *QCoh* is a fpqc stack.

Lemma 4.5. *Let* $F : \underline{\mathrm{Sch}}^{opp} \to$ **Sets** *be an fppf-sheaf and* $\mathcal{X}_F$ *the associated category fibered in groupoids (see Definition 3.5). Then* $\mathcal{X}_F$ *is an fppf-stack.*

Proof. That the Isom functor is a sheaf is easy, since the only morphisms in the fiber categories are the identities. That descent is effective is exactly the sheaf property. □

In particular, the category fibered in groupoids $\mathcal{X}_S$ associated to a scheme S is an fppf-stack, and we will simply write S for it.

The category fibered in groupoids $\mathcal{M}_1$ is not an fppf stack. The problem with genus 1 curve is that the sheaf of differentials Ω^1 and the structure sheaf $\mathcal{O}$ are both of degree zero, so there is no canonical choice of a very ample sheaf. There are examples of Raynaud ([R]) and Zomervrucht ([Z]) of descent data of curves of genus 1 that are not effective. The lack of canonical ample sheaves can be overcome if one considers instead families of curves $C \to S$ admitting sections. We are led to the following definition:

Definition 4.6. An n-pointed curve is the datum of a curve $C \to S$ and of n sections $s_1 \dots, s_n : S \to C$ such that the set-theoretic images $s_1(S), s_2(S), \dots, s_n(S) \subset C$ are disjoint.

Let $g, n \geq 0$ be integers. We denote by $\mathcal{M}_{g,n} \to \underline{\mathrm{Sch}}$ the category fibered in groupoids of n-pointed genus g curves. Its morphisms are morphisms of curves compatible with the n sections.

Fact 4.7. *Let $n \geq 1$, $g \geq 0$. The category fibered in groupoids $\mathcal{M}_{g,n}$ is an fppf stack.*

Idea of the proof. Similar to the proof of Theorem 4.3, except that we use the very ample sheaves $\mathcal{O}((2g+1)s_1)$ in place of the sheaves of differentials. □

Example 4.8.

- $\mathcal{M}_{g,0} = \mathcal{M}_g$;
- the stack $\mathcal{M}_{1,1}$ is the stack of elliptic curves; this is not the j-line $\mathbb{A}^1$, but there is a map of stacks $\mathcal{M}_{1,1} \to \mathbb{A}_1$ through which every map from $\mathcal{M}_{1,1}$ to a scheme factors. The j-line $\mathbb{A}^1$ is an example of *coarse moduli space*.
- there is an equivalence of categories $\mathcal{M}_{0,3} \to \operatorname{Spec} \mathbb{Z}$.
- there is an equivalence of categories $\mathcal{M}_{0,4} \to \mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$.

5 Fiber products in $\underline{\mathrm{CFG}}$

We want to study further the stacks $\mathcal{M}_{g,n}$, so it would be nice if we were able to employ the usual vocabulary of algebraic geometry also when dealing with stacks. For example, we would like to be able to ask ourselves "is the stack $\mathcal{M}_g$ smooth over $\operatorname{Spec} \mathbb{Z}$?", "is the morphism $\mathcal{M}_{g,1} \to \mathcal{M}_g$ surjective?", and so on.

For this, we introduce fiber product in $\underline{\mathrm{CFG}}$. [2]

Definition 5.1. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories fibered in groupoids over $\underline{\mathrm{Sch}}$, and let $\alpha : \mathcal{C} \to \mathcal{D}, \beta : \mathcal{E} \to \mathcal{D}$ be morphisms in $\underline{\mathrm{CFG}}$. The category fibered in groupoids $\mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to \underline{\mathrm{Sch}}$ is defined as follows:

- its objects over a scheme T are of the form (c, e, φ), where c is an object of $\mathcal{C}(T)$, e an object of $\mathcal{E}(T)$ and $\varphi : \alpha(c) \to \beta(e)$ an isomorphism in $\mathcal{D}(T)$.

[2] Actually, the definition we give is that of fiber product in the sub-2-category of $\underline{CFG}$ given by allowing only invertible 2-arrows.

- its morphisms $(c, e, \varphi) \to (c', e', \varphi')$ are given by morphisms $c' \to c'$ in $\mathcal{C}$ and $d \to d'$ in $\mathcal{D}$ over the same $S \to S'$, such that the diagram

$$\begin{array}{ccc} \alpha(c) & \xrightarrow{\varphi} & \beta(e) \\ \downarrow & & \downarrow \\ \alpha(c') & \xrightarrow{\varphi'} & \beta(e') \end{array}$$

  commutes.

Example 5.2. Let $\mathcal{M}_g \to \mathcal{M}_g \times \mathcal{M}_g$ be the diagonal map, and let $T \to \mathcal{M}_g \times \mathcal{M}_g$ correspond to a pair (C, C') of curves over T. Then the fiber product $\mathcal{M}_g \times_{\mathcal{M}_g \times \mathcal{M}_g} T$ is the sheaf $\mathrm{Isom}_T(C, C')$ (seen as a category fibered in setoids).

Definition 5.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of categories fibered in groupoids over Sch. We say that f is *representable by schemes* if for every scheme T and morphism $T \to \mathcal{D}$ in CFG, the fiber product $\mathcal{C} \times_{\mathcal{D}} T$ is (equivalent to) a scheme.

Definition 5.4. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of categories fibered in groupoids over Sch representable by schemes. Let $\mathcal{P}$ be a property of morphisms of schemes that is:

- stable under base change;
- fppf local on the target.

We say that f has the property $\mathcal{P}$ if for all schemes T and morphisms $T \to \mathcal{D}$, the base change $\mathcal{C} \times_{\mathcal{D}} T \to T$ has the property $\mathcal{P}$.

Example 5.5. Let T be a scheme, g a non-one non-negative integer, and let $T \to \mathcal{M}_g$ correspond to a curve $C \to T$ in $\mathcal{M}_g(T)$. Then there is a cartesian diagram in CFG

$$\begin{array}{ccc} C & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,1} & \longrightarrow & \mathcal{M}_g \end{array}$$

where the lowest arrow is the forgetful morphism. So $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is smooth, proper, surjective, of relative dimension 1. This earns the morphism of stacks $\mathcal{M}_{g,1} \to \mathcal{M}_g$ the name of *universal curve* of genus g. Indeed, any curve $C \to T$ arises as a pullback of $\mathcal{M}_{g,1} \to \mathcal{M}_g$!

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