The algebraic stack $\mathcal{M}_g$

Raymond van Bommel

9 October 2015

The author would like to thank Bas Edixhoven, Jinbi Jin, David Holmes and Giulio Orecchia for their valuable contributions.

1 Fibre products and representability

Recall from last week ([G]) the definition of categories fibred in groupoids over a category $\mathcal{S}$. Besides objects there are 1-morphisms (functors) between objects and 2-morphisms (natural transformations (that are isomorphisms automatically)) between the 1-morphisms. They form a so-called 2-category, $\underline{\text{CFG}}$. We also defined the fibre product in $\underline{\text{CFG}}$. In this chapter we will take a general base category $\mathcal{S}$ unless otherwise indicated. However, if wanted, you can think of $\mathcal{S}$ as being $\underline{\text{Sch}}$.

Given an object $S \in \text{Ob}(\mathcal{S})$, we can consider it as category fibred in groupoids, which is also called $\mathcal{S}/S$, by taking arrows $T \to S$ in $\mathcal{S}$ as objects and arrows over S in $\mathcal{S}$ as morphisms. The functor $\mathcal{S}/S \to \mathcal{S}$ is the forgetful functor, forgetting the map to S.

Exercise 1. Let $S, T \in \text{Ob}(\mathcal{S})$. Check that the fibre $(\mathcal{S}/S)(T)$ is isomorphic to the set $\text{Hom}(T, S)$, i.e. the category whose objects are the elements of $\text{Hom}(T, S)$ and whose arrows are the identities.

Proposition 2. *Let $\mathcal{X}$ be a category fibred in groupoids over $\mathcal{S}$. Then $\mathcal{X}$ is equivalent to an object of $\mathcal{S}$ (i.e. there is an equivalence of categories that commutes with the projection to $\mathcal{S}$) if and only if $\mathcal{X}$ has a final object.*

Proof. Given an object $S \in \text{Ob}(\mathcal{S})$ the associated category fibred in groupoids has final object $\text{id}\colon S \to S$.

On the other hand, if $\mathcal{X}$ has a final object lying over $S \in \mathrm{Ob}(\mathcal{S})$, then it is an easy exercise, using the properties of categories fibred in groupoids, to prove that it is equivalent to $\mathcal{S}/S$. $\square$

Now let us repeat the definition of the fibred product.

Definition 3 ([G])**.** Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories fibred in groupoids over $\mathcal{S}$, and let $\alpha\colon \mathcal{C} \to \mathcal{D}$ and $\beta\colon \mathcal{E} \to \mathcal{D}$ be 1-morphisms in $\underline{\mathrm{CFG}}$. Then the *fibred product* $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ is defined as follows:

- its objects over $T \in \mathcal{S}$ are triples (c, e, φ), where $c \in \mathrm{Ob}(\mathcal{C}(T))$, $e \in \mathrm{Ob}(\mathcal{E}(T))$ and $\varphi\colon \alpha(c) \to \beta(e)$ is an isomorphism in $\mathcal{D}(T)$;
- its morphisms $(c, e, \varphi) \to (c', e', \varphi')$ are pairs of morphisms $f\colon c \to c'$ and $g\colon e \to e'$ over the same $S \to S'$ such that the diagram

$$\begin{tikzcd} \alpha(c) \arrow[r, "\varphi"] \arrow[d, "\alpha(f)"'] & \beta(e) \arrow[d, "\beta(g)"] \\ \alpha(c') \arrow[r, "\varphi'"] & \beta(e') \end{tikzcd}$$

  commutes.

Example 4. Take $\mathcal{S} = \underline{\mathrm{Sch}}$. Let $S \in \mathrm{Ob}(\underline{\mathrm{Sch}})$ and $T, V \in \mathrm{Ob}(\underline{\mathrm{Sch}}/S)$. Then S, T and V can be considered as objects of $\underline{\mathrm{CFG}}$ by taking $(\underline{\mathrm{Sch}}/S), (\underline{\mathrm{Sch}}/T)$ and $(\underline{\mathrm{Sch}}/V)$, respectively. Then the product $T \times_S V$ in $\underline{\mathrm{Sch}}$ is canonically isomorphic to the product $T \times_S V$ in $\underline{\mathrm{CFG}}$.

Remark 5. First consider the bottom part of this diagram.

$$\begin{tikzcd} \mathcal{T} \arrow[dr, dashed, "{(c,e)}"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "c"'] & & \\ & \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \arrow[r, "p_{\mathcal{E}}"] \arrow[d, "p_{\mathcal{C}}"] & \mathcal{E} \arrow[d, "\beta"'] \\ & \mathcal{C} \arrow[r, "\alpha"] & \mathcal{D} \end{tikzcd}$$

Remark that it is not a commutative diagram in the classical sense. The 1-morphisms $\beta \circ p_{\mathcal{E}}$ and $\alpha \circ p_{\mathcal{C}}$ are not equal, but there is a 2-morphism between them, which should be considered as part of the data of the diagram.

Now the fibred product has the following universal property. For diagrams as above, i.e. for every $\mathcal{T} \in \mathrm{Ob}(\underline{\mathrm{CFG}})$, 1-morphisms $e\colon \mathcal{T} \to \mathcal{E}$ and $c\colon \mathcal{T} \to \mathcal{C}$, and 2-morphism $\alpha \circ c \Rightarrow \beta \circ e$, there exists a unique 1-morphism $(c, e)\colon \mathcal{T} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ such that $p_{\mathcal{C}} \circ (c, e) = c$ and $p_{\mathcal{E}} \circ (c, e) = e$, and the 2-morphism $\alpha \circ c \Rightarrow \beta \circ e$ is induced by the 2-morphism $\alpha \circ p_{\mathcal{C}} \Rightarrow \beta \circ p_{\mathcal{E}}$.

Exercise 6. Let $\mathcal{S} = \underline{\text{Sch}}$ and $n \in \mathbb{N}_0$. Consider the diagonal $\Delta\colon \mathcal{M}_g \to \mathcal{M}_g \times \mathcal{M}_g$ (which means $\mathcal{M}_g \times_{\mathcal{S}} \mathcal{M}_g$). For $S \in \mathrm{Ob}(\underline{\text{Sch}})$ a 1-morphism $S \to \mathcal{M}_g \times \mathcal{M}_g$ corresponds to an ordered pair (C, C') of genus g curves over S. Prove that the following diagram is cartesian in the 2-categorical sense:

$$\begin{tikzcd}
\mathfrak{Isom}_S(C,C') \arrow[r, "s"] \arrow[d, "C"] & S \arrow[d, "{(C,C')}"] \\
\mathcal{M}_g \arrow[r, "\Delta"] & \mathcal{M}_g \times \mathcal{M}_g
\end{tikzcd}$$

where the 2-morphism $\Delta \circ C \Rightarrow (C, C') \circ s$, i.e. the functorial isomorphism of pairs $(C, C) \simeq (C, C')$, is given by $(\varphi \mapsto T) \mapsto \eta_\varphi := (\mathrm{id}_{C_T}, \varphi)$.

Exercise 7. Let $f\colon \mathcal{M}_{g,1} \to \mathcal{M}_g$ be the forgetful 1-morphism, let $S \in \mathrm{Ob}(\underline{\text{Sch}})$ and let $S \to \mathcal{M}_g$ be a 1-morphism corresponding to a curve C/S. Prove that there is a natural cartesian diagram (in the 2-categorical sense) as below. Describe all 1-morphisms and 2-morphisms which are not indicated.

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & \mathcal{M}_{g,1} \arrow[d, "f"] \\
S \arrow[r, "C"] & \mathcal{M}_g
\end{tikzcd}$$

This morphism f is also called the *universal curve* over $\mathcal{M}_g$ as any genus g curve fits in such a diagram.

Exercise 8. The following funny fact might be a nice exercise for the categorical reader: prove that it suffices to check the universal property only for test objects $\mathcal{T}$ which are objects of $\mathcal{S}$.

Definition 9 ([G])**.** Let $f\colon \mathcal{C} \to \mathcal{D}$ be a 1-morphism in $\underline{\text{CFG}}$. We say that f is *representable (by objects of $\mathcal{S}$)* if for every $T \in \mathrm{Ob}(\mathcal{S})$ and every 1-morphism $T \to \mathcal{D}$ in $\underline{\text{CFG}}$, the fibre product $\mathcal{C} \times_{\mathcal{D}} T$ is equivalent to an object of $\mathcal{S}$.

Definition 10 ([G])**.** Let $f\colon \mathcal{C} \to \mathcal{D}$ be a representable 1-morphism in $\underline{\text{CFG}}$ over $\mathcal{S} = \underline{\text{Sch}}$. Let $\mathcal{P}$ be a property of morphisms of schemes that is stable under base change and fppf-local on the target. Then we say that f *has property* $\mathcal{P}$ if for all $T \in \mathrm{Ob}(\mathcal{S})$ and all 1-morphisms $T \to \mathcal{D}$ the base case change $\mathcal{C} \times_{\mathcal{D}} T \to T$ has property $\mathcal{P}$.

Remark 11. One could argue that it is not necessary to assume the fppf-locality at this point. This is true, but we will use this mainly for stacks over $\underline{\text{Sch}}$ with the fppf-topology and then we want the notion to be compatible with the descent.

Example 12. If $S \to T$ is a morphism in $\underline{\text{Sch}}$ having a property $\mathcal{P}$ as above, then the associated 1-morphism of $\underline{\text{CFG}}$ over $\mathcal{S} = \underline{\text{Sch}}$ is representable and has property $\mathcal{P}$.

Example 13. Let $g \geq 2$. From exercise 7 we can conclude that the forgetful 1-morphism $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is smooth of relative dimension 1, proper and surjective.

Proposition 14 ([V]). *Let $\mathcal{X}$ be a $\underline{\text{CFG}}$ and let $\Delta\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ be the diagonal morphism. Then Δ is representable if and only if for every $T \in \mathrm{Ob}(\mathcal{S})$ every 1-morphism $T \to \mathcal{X}$ is representable.*

Proof. Suppose that Δ is representable. Then we want to show that any 1-morphism $f\colon T \to \mathcal{X}$ is representable. To do this, take an arbitrary 1-morphism $g\colon S \to \mathcal{X}$ and consider $S \times_{\mathcal{X}} T$. Using the universal property we can verify that the following diagram is cartesian.

$$\begin{tikzcd} S \times_{\mathcal{X}} T \arrow[r] \arrow[d] & S \times_{\mathcal{S}} T \arrow[d, "g\times f"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_{\mathcal{S}} \mathcal{X} \end{tikzcd}$$

Then the representability of Δ implies that $S \times_{\mathcal{X}} T$ is represented by a scheme.

Now suppose that every 1-morphism $T \to \mathcal{X}$ is representable. Now suppose that $S \in \mathrm{Ob}(\mathcal{S})$ and that $(f,g)\colon S \to \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ is a 1-morphism. Then we have the following diagram.

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{X}\times_{\mathcal{S}}\mathcal{X}} S \arrow[r] \arrow[d] & S \times_{f,\mathcal{X},g} S \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ S \arrow[r, "\Delta_S"] & S \times_{\mathcal{S}} S \arrow[r, "f\times g"] & \mathcal{X} \times_{\mathcal{S}} \mathcal{X} \end{tikzcd}$$

The right and the big square are cartesian, hence the left square is cartesian. Now $\mathcal{X} \times_{\mathcal{X}\times_{\mathcal{S}}\mathcal{X}} S$ is a scheme as $S, S \times_{\mathcal{S}} S$, and $S \times_{f,\mathcal{X},g} S$ are schemes by assumption. $\square$

Example 15. Let $n \in \{4,5\}$. Take $\mathcal{S} = \underline{\text{Sch}}/\operatorname{Spec}(\mathbb{Z}[\frac{1}{n}])$. Consider the functor $\mathcal{C}$ which associates to a scheme S the set of pairs of an elliptic curve over S and a section everywhere of order n. This is a category fibred in groupoids. Consider the forgetful 1-morphism $f\colon \mathcal{C} \to \mathcal{M}_{1,1}$. This morphism is representable, étale, finite of degree 12 resp. 24, as for elliptic curves E/S the morphism $(E[n] \setminus E[2]) \to S$ has these properties if $\operatorname{char} S \nmid n$.

2 Algebraic stacks

In this chapter we will take $\mathcal{S} = \underline{\mathrm{Sch}}$ and $g \geq 2$. We will introduce the notion of an algebraic stack over $\mathcal{S}$, with the fppf-topology, and prove that $\mathcal{M}_{g,n}$ is an algebraic stack.

Proposition 16. *The diagonal map $\Delta\colon \mathcal{M}_{g,n} \to \mathcal{M}_{g,n} \times \mathcal{M}_{g,n}$ is representable by schemes, proper and unramified.*

Proof. By proposition 14 we need to check that every 1-morphism f from a scheme T to $\mathcal{M}_{g,n}$ is representable. Let C/T be the curve corresponding to f. Let $S \in \mathrm{Ob}(\underline{\mathrm{Sch}})$ an abitrary scheme and let $g\colon S \to \mathcal{M}_{g,n}$ be a 1-morphism corresponding to a curve C'/S. Then we need to check that $S \times_{\mathcal{M}_{g,n}} T$ is equivalent to a scheme. By exercise 6, the latter is $\mathfrak{Isom}_{T\times S}(C_{T\times S}, C'_{T\times S})$. As the genus is at least 2, there are canonical polarisations on C and C'. Then a result of Grothendieck implies that this $\mathfrak{Isom}$-sheaf is represented by a scheme, see for example [DM, p. 84] or [N, p. 31].

To check that Δ is unramified, you need to check that $\mathfrak{Isom}_{T\times S}(C_{T\times S}, C'_{T\times S})$ is unramified. To do this we can reduce to the case of an algebraically closed field k and check that $\mathfrak{Aut}_k(C)$ is unramified. Then $k[\varepsilon]/(\varepsilon^2)$-points correspond to vector fields on C and there are no non-zero vector fields over smooth genus g curves. In other words, $\mathfrak{Aut}_{k[\varepsilon]/(\varepsilon^2)}(C_{k[\varepsilon}/(\varepsilon^2)) \to \mathfrak{Aut}_k(C)$ is a bijection. Note that we use that $g \geq 2$.

To prove that Δ is proper, we use the valuative criterium for properness, which amounts to the following geometric theorem. More details can be found in [DM, p. 84]. ☐

Theorem 17 ([DM, Lemma I.12])**.** *Let X and Y be smooth curves of genus g over a discrete valuation ring R with algebraically closed residue field. Then any isomorphisms between the generic fibres X_η and Y_η extends uniquely to an isomorphism between X and Y.*

Remark 18. In fact, now we have also proved the finiteness of the diagonal. In particular, this implies that $\mathfrak{Aut}_{\overline{k}}(C)$ is finite for any genus g curve C over an algebraically closed field $\overline{k}$.

Exercise 19. Prove that theorem 17 is not true for $g = 0$.

Theorem 20. *There exists a scheme H and a 1-morphism $H \to \mathcal{M}_{g,n}$ that is surjective and smooth.*

Proof. Details for this proof can be found in [DM, p. 78]. As we saw in [G], there is a canonical embedding for any genus g curve $f\colon C \to S$ in $\mathbb{P}(f_*(\Omega^1_{C/S}{}^{\otimes 3}))$. Now $f_*(\Omega^1_{C/S}{}^{\otimes 3})$ is free of rank $5g-6$. We will consider the functor

$$H\colon S \mapsto \{(f,\varphi) : f\colon C \to S \text{ of genus } g \text{ and } \varphi\colon \mathbb{P}(f_*(\Omega^{\otimes 3}_{C/S})) \xrightarrow{\sim} \mathbb{P}^{5g-6}_S\}.$$

On the one hand, H is represented by an open subscheme of the Hilbert scheme of $\mathbb{P}^{5g-6}$, which represents the set of closed subschemes of $\mathbb{P}^{5g-6}$ that are flat and of finite presentation over the base. On the other hand, there is an action of PGL_{5g-5} on H by postcomposing φ with an automorphism of $\mathbb{P}^{5g-6}$. Now $\pi\colon H \to \mathcal{M}_{g,n}$ is smooth and surjective, as for each curve $f\colon C \to S$ corresponding to $S \to \mathcal{M}_{g,n}$ the fibre product with π is represented by the scheme of isomorphisms between $\mathbb{P}^{5g-6}_S$ and $\mathbb{P}(f_*(\Omega^{\otimes 3}_{C/S}))$, which is smooth and surjective over S. $\square$

Remark 21. To get a coarse moduli space $M_{g,n}$, you take the geometric invariant theory quotient (also known as the categorical quotient in $\underline{\mathrm{Sch}}$, which is given on affines by taking the ring of invariants) of H by PGL_{5g-5}. This coarse moduli has the defining property that every morphism from $\mathcal{M}_{g,n}$ to a scheme S factors uniquely through $M_{g,n}$.

Theorem 22 ([DM, Theorem 4.21])**.** *Let $\mathcal{X}$ be a stack over $\underline{\mathrm{Sch}}$. Assume that the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, unramified and proper. Moreover, assume that there exists a scheme H of finite type over $\mathbb{Z}$ and a smooth surjective morphism $H \to \mathcal{X}$. Then there exists a scheme S and an étale surjective morphism $S \to \mathcal{X}$.*[1]

Now we managed to prove that $\mathcal{M}_{g,n}$ is an algebraic stack even before knowing what this means.

Definition 23. Let $\mathcal{X}$ be a stack over $\underline{\mathrm{Sch}}_{\mathrm{fppf}}$. Then $\mathcal{X}$ is called a *Deligne-Mumford stack* if the diagonal of $\mathcal{X}$ is representable and there exists a scheme S together with a surjective and étale morphism $S \to \mathcal{X}$.

Remark 24. The first condition that the diagonal is representable is necessary for the second condition to even make sense, as it implies that $S \to \mathcal{X}$ is representable.

Remark 25. There is also the weaker notion of an *Artin stack* or *algebraic stack* in the literature. Here you only require the diagonal to be represented by an algebraic space[2] and the morphism $S \to \mathcal{X}$ to be only smooth and surjective. Note that in the past people have also used called Deligne-Mumford stacks algebraic stacks.

Example 26. Also $\mathcal{M}_{1,n}$ for $n \geq 1$ and $M_{0,n}$ for $n \geq 3$ are Deligne-Mumford stacks, and $\mathcal{M}_{0,n}$ for $n \in \{0, 1, 2\}$ is an Artin stack. You can use a similar strategy using other sheafs to get canonical embeddings, see also [G].

For $\mathcal{M}_{1,1}$ we can also give a direct proof. The proof of the representability of the diagonal is similar to the proof given in proposition 16, but the canonical polarisation is given by the marked point, instead of by $\Omega^{\otimes 3}$. The cover by a scheme is given by the two 1-morphisms in example 15: the forgetful 1-morphism from elliptic curves over $\mathbb{Z}[\frac{1}{2}]$ with a point of order 4, covering $(\mathcal{M}_{1,1})_{\mathbb{Z}[1/2]}$, and the forgetful 1-morphism from elliptic curves over $\mathbb{Z}[\frac{1}{5}]$ with a point of order 5, covering $(\mathcal{M}_{1,1})_{\mathbb{Z}[1/5]}$. The fact that these two stacks are equivalent to a scheme can be found in [CE].

[1] Here you really start needing the fppf-topology.

[2] An algebraic space is a sheaf over the fppf-site such that the diagonal is representable by schemes and there exists an étale surjective cover from a scheme. However, this difference is irrelevant. In fact, the 'correct' definition of a Deligne-Mumford stack should use algebraic spaces.

3 Quotient stack

In this section we will define the quotient of a scheme by a group as stack. In the setup we have a scheme X over a scheme S and an affine smooth[3] group scheme G of finite type over S, operating on X.

Definition 27 ([DM, Ex. 4.8])**.** The *quotient stack* $[X/G]$ over $\underline{\mathrm{Sch}}$ is defined as follows. For a scheme $T \in \mathrm{Ob}(\underline{\mathrm{Sch}})$ the objects of $[X/G]$ are G-torsors E over T (i.e. schemes E over T with a right action of G_T such that on some fppf-cover $V \to T$ we have $E \times_T V \cong G_V$ as scheme with an action of G_V) together with a G-equivariant map $\varphi \colon E \to X$. An arrow from $E \to T$ to $E' \to T'$ is the data of a cartesian diagram

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d] & E' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

such that ψ is G-equivariant and $\varphi = \varphi' \circ \psi$.

Remark 28. An example of an object over X in $[X/G]$ is $G \times X$, where G acts on the left and $\varphi \colon X \times G \to X$ is the projection on the first coordinate. This object gives rise to a morphism $X \to [X/G]$ of stacks that is smooth and surjective. You can use this to prove that $[X/G]$ is an Artin stack. Moreover, the morphism $X \to [X/G]$ is the *universal G-torsor over X*, i.e. every G-torsor $Y \to T$ with a G-equivariant map $Y \to X$ arises as a base change of $X \to [X/G]$.

Example 29. In the previous section we saw how $M_{g,n}$ can be realised as the quotient stack $[H/\mathrm{PGL}_{5g-5}]$.

Exercise 30 ([LOOV, p. 45–47])**.** Let $\mathbb{G}_m$ act on $X := \bigcup_{0 \le i \le n} D(x_i) \subset \mathbb{A}^{n+1}$ by multiplication. Prove that $[X/G] = \mathbb{P}^n$.

Example 31. If you take $X = S$, you get the stack classifying torsors for a group scheme. This stack is also called the *classifying stack* of G, also denoted by BG.

Exercise 32. Prove that $M_{0,0} \cong B\mathrm{PGL}_2$.

[3]In fact you could take G to be étale and separated instead of affine smooth and then the resulting quotient stack is a Deligne-Mumford stack.

References

[CE] J. Couveignes, B. Edixhoven, *Computational Aspects of Modular Forms and Galois Representations.* Available at: http://www.math.u-bordeaux1.fr/~jcouveig/book.htm.

[DM] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus.* Publications Mathématiques de l'IHÉS 36: 75–109.

[LOOV] M. Lieblich, M. Olsson, B. Osserman, R. Vakil. *Deformations and Moduli in Algebraic Geometry.* Available at: https://www.math.ucdavis.edu/~osserman/math/Defbookactive.pdf.

[N] N. Nitsure (et al.), *FGA explained. Construction of Hilbert and Quot schemes.* Math. surveys and monographs 123, AMS, 2005.

[G] G. Orecchia, *The moduli stack* $\mathcal{M}_g$. Available at: http://pub.math.leidenuniv.nl/~bommelrvan/2015/Mg1_Orecchia.pdf.

[S] The Stacks Project authors, *The Stacks Project.* Available at: http://stacks.math.columbia.edu/.

[V] J. Voight, *Introduction to algebraic stacks.* Available at: https://math.dartmouth.edu/~jvoight/notes/moduli-red-harvard.pdf.