Some very basic deformation theory

Carlo Pagano

W We warm ourselves up with some infinitesimal deformation tasks, to get a feeling of the type of question we are dealing with:

1 Infinitesimal Deformation of a sub-scheme of a scheme

Take k a field, X a scheme defined over k, Y a closed subscheme of X. An infinitesimal deformation of Y in X, for us, is any closed subscheme $Y' \subseteq X \times_{spec(k)} spec(k[\epsilon])$, flat over $k[\epsilon]$, such that $Y' \times_{Spec(k[\epsilon])} Spec(k) = Y$. We want to get a description of these deformations. Let's do it locally, working the affine case, and then we'll get the global one by patching. Translating the above conditions you get: classify ideals $I' \subseteq B' := B[\epsilon]$, such that I' reduces to I modulo ϵ and B'/I' is flat over $k[\epsilon]$. Flatness over $k[\epsilon]$ can be tested just looking at the sequence $0 \to (\epsilon) \to k[\epsilon] \to k \to 0$(observe that the first and the third are isomorphic as $k[\epsilon]$-modules), being so equivalent to the exactness of the sequence $0 \to B/I \to B'/I' \to B/I \to 0$. So in this situation you see that if you have such an I' then every element $y \in I$ lift uniquely to an element of B/I, giving rise to a map of $B - modules$, $\varphi_I : I/I^2 \to B/I$, proceeding backwards from $\varphi' \in Hom_B(I, B/I)$, you gain an ideal I' of the type wanted(to check this use again the particularly easy test of flatness over $k[\epsilon]$). So we find that our set of deformation is naturally parametrized by $Hom_B(I/I^2, B/I)$. Patching together one get that the set of deformations is naturally parametrized by global sections of $Hom_Y(I/I^2, O_Y)$, where I is the sheaf of ideals defining Y as a closed subscheme of X. It is the normal bundle of Y in X, and denoted by $N_{Y|X}$(the terminology comes from assuming everything non-singular it is the tangent bundle of X restricted to Y quotiented out by the tangent bundle of Y), and so denote the parametrization to our deformation problem as $H^0(Y, N_{Y|X})$.

2 Infinitesimal Deformation of a line bundle on a scheme

Let X, X', k as in the previous section, let L be a line bundle on X.We want to study line bundles in X' that restricts to L from the reduction map mod ϵ. (That is we want to study line bundles L' on X' such that $L' \otimes O_X \simeq L$). But the datum of a line bundle is expressible in terms of cocycles in $H^1(X', O_X'^*)$ and $H^1(X, O_X^*)$ respectively, and we may rephrase our question as asking for a cocycle sitting in the first group that reduces to a given cocycle in the second. But for a ring being a unit or being a unit modulo nilpotents is the same, thus(patching) one gets $O^*_{X'} = O^*_X + \epsilon O_X$. Thus the lifting of a given cocycle comes from a free choice of a cocycle living in $H^1(X, O_X)$. So we get that $H1(X, O_X)$ parametrizes deformations of a given line bundle on X(in a totally explicit way in terms of the cocycles). (Maybe it is natural at this point to think about a Riemann surface, where from the exponential sequence you have a sequence $H^1(X, \mathbb{Z}) \to H^1(X, O_X) \to H^1(X, O^*_X) \to \mathbb{Z} \to 0$, where the last map is the degree of a line bundle, so one consistently see that one can perform continuous deformation of a line bundle only inside a given coset of the degree map that is indeed gained by $H^1(X, O_X)$ modulo a discrete subgroup, or on the other extreme to the case of a $K3$ surface where $H^1(X, O_X) = 0$, and where indeed the Pic comes as a lattice where there is no infinitesimal deformation at all).

3 Infinitesimal Deformation of a Scheme

This looks a bit harder than the previous 2, and it is actually the infinitesimal version of the question we are facing(in the rest of the seminar):one reason is that we do not immediately have an ambient setup where to define the set we are interested in, so first thing we fix this. We are basically interested in performing the, first order, infinitesimal step in constructing a flat family around our favourite scheme X, say, over a field k. So it is reasonable to say that our data is a couple (X', i) with X' a flat scheme over $k[\epsilon]$, and $i' : X \to X'$ a closed immersion inducing an isomorphism between X and $X' \times_A Spec(k)$(in other words an isomorphism with the fiber of X' over the closed point via the structural map on $Spec(k[\epsilon])$). Morphism are defined in the obvious way with the requirement of respecting also the data of the map i. It is not clear how to proceed in understanding this data, but, as we wrote in line 4 of this section, it is clear what we want to do. So let us see what happens in a context where we can easily define loads of family and do computations on them(so to not feel anymore the difficulty described at line 2 and 3), to get a better feeling of the task:

3.1 Infinitesimal Deformation of a Complex Manifold

The basic idea:Take M a compact complex manifold of dimension n. Then it is basically given by a disjoint union of trivial pieces(say polydiscs) and a cocycle of holomorphic identifications $f_{i,j}(z)$, so we can think of applying deformations to M by applying deformations to the above cocycles while keeping the same set of polydiscs. In other words to write down $F_i, j(z, t)$ with $F_{i,j}(z, 0) = f_{i,j}(z)$, with F having a smooth dependence on t, so that in order to see how the complex structure changes we can differentiate the glueing condition with respect to t. From this last operation we want to, for instance recognize, trivial families, but there is no reason to expect that this comes from a naiv vanishing of the differentiation because one may recognize a trivial family only after an isomorphism with a trivial family. So in order to be able to even speak about isomorphism of families let us make a little bit of setup on the notion of family.

Definition:
Let B an open connected interval in $\mathbb{R}$. A differentiable family of compact complex manifolds over B is the data (S, f) a smooth manifold S with a smooth surjective

map $f : S \to B$, wich is
1)non-singular,
2)$f^{-1}(t)$ is a compact connected subset of S
3) you have locally finite open covering of S, $\{U_{hh \in H}\}$ and a subordinate family of n smooth complex valued functions $z_{1,h}, ..., z_{n,h}$ each of them from U_h to $\mathbb{C}$, such that for all $t \in B$ the system $\{U_i \bigcap f^{-1}(t)\}_{h \in H}$, and is a complex atlas via the coordinate map $\{(z_1, ..., z_n, h)\}_{h \in H}$.
With this definition we have an obvious notion of morhpism of differentiable family of compact complex manifolds over B, and a notion of trivial family(as one isomorphic to a product family $M \times B$ for some compact complex manifold M).
The definition is made in order to do the computation outlined in the Basic idea: you can check(via the implicit function theorem) how, locally in the parameter space B, we can think our family as disjoint union of polyidiscs with identifications(and by 3) locally finite each of them intersect only finitely many others) $\{f_{i,j}(z,t)\}_{(i,j) \in H^2}$ holomorphic in the first coordinate and smooth respect to the second (that is now a subinterval of B small enough). The identification enjoy a cocycle condition that as an exercise you may like to differentiate respect to t and get, from carefully applying the chain rule, in a given point t_0 a collection of vector fields $\{\theta_{i,j}\}_{(i,j) \in H^2}$, with $\theta_{i,j} \in H^0(M_{t_0} \bigcap U_{i,j}, T_{M_{t_0}})$(we mean the tangend bundle), wich satisfies $\theta_{i,j} + \theta_{j,k} = \theta_{i,k}$(this relation is coming from the one satisfied by the $f_{i,j}$). In other words you get an element of $H^1(M_{t_o}, T_{M_{t_0}})$. You can as well check that the cohomology class doesn't change for refinement of the open neighbour and it doesn't change by changing local coordinates and so giving an isomorphism of family(locally around the point you are differentiating). So you call $\frac{d}{dt}((S,f))_{t_0}$, this cohomology class, and so you find that for trivial families is constantly 0 (In the book [2] you can see this computation done in detail). Under some condition on the family is possible also to prove partial converse to this result((for more info look at [2]).

So now we have a clear candidate for parametrizing the deformations asked at the beginning of this section: $H^1(X, T_X)$.
And indeed this will be the answer for a smooth variety over k. But let's observe that now a new question is immediately clear from the context of deformations of a complex variety: we started with a family and we looked at the behaviour around a point at the first order(we literally saw this in the process of differentiating the identification conditions at a given point), and we got an infinitesimal deformation living in $H^1(X, T_X)$, but it is likely that we are interested in the converse:
Question: given a cohomology class $\theta \in H^1(X, T_X)$ can I put my X in a 1-dimensional family so that θ is the corresponding infinitesimal deformation at X?

3.2 Back to schemes

If one tries to abstract what are the features of the computation of the previous section that lead to the above mentioned result, may find the following:
Locally a manifold is a union of trivial objects, and by the definition of a differentiable family (the implicit function theorem and a bit of work) deformations are locally(taking an open cover of our variety small enough) trivial, expressing the

various trivialization that one gets a cocycle in the automorphism of each overlap, wich at the first order gave a cocycle in the tangent sheaf.
A scheme is a union of trivial pieces as well(the affine pieces), so by we are lead immediately to the following question:
Question: Are first order deformations(as defined at the beginning of this section) of an affine smooth scheme over k, always trivial?

The answer is yes. One way to see it is rephrasing smoothness differently by:
Theorem(Infinitesimal lifting property)
Let X be an affine smooth scheme of finite type over k. Let Y be an affine scheme over k, let $Y \subseteq Y'$ a closed embedding with associated sheaf of ideals nilpotent(is said an infinitesimal thickening). Then every morphism $f : Y \to X$ extends to a morphism $Y' \to X$. The converse is true: if the above property holds for every Y, Y', f as before, then X is smooth.

For a self contained proof of this fact look at Hartshorne[1]. Let me also comment that in the course of the proof you hit the following exact sequence of B-modules(B being the coordinate ring of your smooth scheme, and B a finite polynomial ring gained by writing A as B/I)
$0 \to I/I \to \Omega^1_{A/k} \otimes_A B \to \Omega_{B/k} \to 0$, where, being $Spec(B)$ smooth, $\Omega_{B/k}$ is projective, then giving a splitting of the above sequence wich in turns give an epimorphism $Hom(\Omega_{A/K}, A) \to Hom(I/I^2, A) \to 0$. And in fact one can prove that, in the affine case but dropping the smoothness assumption, the cokernel of the last map is in natural bijection with the set of deformation of $Spec(A)$, proving again that in the smooth case there are non trivial deformations (you can learn a more general setting for facing deformation probelms in [1], where the T^i functors are introduced). (It is interesting to compare this cokernel with the output of the first section, being actually a quotient of the normal bundle in the affine space that was parametrizing deformations of $Spec(A)$ as a subscheme of the affine space $Spec(B)$.Is there a quick way of concluding directly that to obtain deformations of $Spec(A)$ one has to mod out the previous deformations(as a subscheme of $Spec(B)$) by the image of the above map?).

With this result we can proceed along the lines of the beginning of this subsection:
Take now X a smooth variety over k . Consider X' an infinitesimal deformation of X. Take an affine covering of X by U_i, take the corresponding covering U'_i of X'. Thus U'_i as infinitesimal deformations of U_i are trivial choose a trivialization $\varphi_i : U_i \times_{Spec(k)} Spec(k[\epsilon]) \to U'_i$ for each i, thus $\varphi_j^{-1}\varphi_i$ gives a cocycle in a sheaf of automorphism group, wich we now want to prove being just the tangent sheaf. Indeed this comes down to the following algebraic statement(we are on affine opens):
Let A be a smooth affine algebra over k then automorhpsims of $A[\epsilon]$ as $k|\epsilon]$-algebra, congruent the identity modulo ϵ are canonically isomorphic to $Hom_A(\Omega^1_{A/k}, A)$, or, in more geometric language, with $H^0(Spec(A), T_{(Spec(A))})$. (If you are not caring about a canonical argument, you can quickly convince your self expressing A as a polynomial algebra over k and find out that you are only allowed to add to each variable a multiple of ϵ in A in a way that the corresponding polynomial relations of the ideal defining A are preserved, expanding the polynomial everything dies except the first order terms wich satisfy a relation that is exactly the one

given by $\Omega^1_{A/k}$ once you express him in coordinates so the choiche of a value of A for each coordinate under thhese constraints correspond exactly to an element of $Hom_A(\Omega^1_{A/k}, A)$. But we need to be sure our argument is canonical because we want to patch the various isomorphism to an isomorphism of sheafs).
Here is the bijection:given $\varphi \in Hom_A(\Omega^1_{A/k}, A)$, consider $a \to a + \epsilon(\varphi(da))$, conversely $a \to \tau(a) - a \in \epsilon A$, is a derivation so gives an element in $Hom_A(\Omega^1_{A/k}, A)$. Thus what we got is a cocycle in $H^1(X, T_X)$, easily seen independent of the covering chosen. Conversely starting with such a cocycle, you build up automorphisms by the previous argument, that gives you (trivial) deformations on each open of a covering, that glue togheter to an infinitesimal deformation of X. The two procedure give a natural bijection between the set of infinitesimal deformation of X and $H^1(X, T_X)$.

4 Functor of Artin rings, and (formal) smoothness

We now want to deal with the question we hit in section 3.2. In the [reference to Kodaira] you find that in the complex analytic case there is an obvious obstruction for a cohomology class to be the tangent vector of a 1-dimensional family, and this obstruction lives in $H^2(X, T_X)$. Our case will be analogous. A clear difference between the 2 setup is that in our case we could speak easily of an infinitesimal family in term of an actual family over a thickening of $Spec(k)$, so it is clear that the next step is to keep going on to higher and higher thickenings. It is clear that before we were starring at just two values of a whole functor, $F(k)$ our object(one of the 3 previous section) and the tangent space at it for our moduli problem $F(k[\epsilon])$. So in our case a first step in putting our first order approximations, living in $F(k[\epsilon])$, in a family would be to extend them to arbitrarily thickened Artin ring. So we now look at covariant functors $F : \{local - artin - ring - over - k\} \to Set$, such that $F(k)$ has only one element(the object you would like to put in family). Given the domain it is clear that we can relax the concept of representability of these functors at least to the concept of being represented by a complete local Noetherian ring(complete with respect to the topology induced by the maximal ideal). It turns out that regularity of these rings is equivalent to the following(this is basically what we have seen in the criterion of smoothness via the infinitesimal lifting property): given $A' \to A$ a morhpism of local artin ring over k, and $O \to A$ a morhpism of k-algebras, one can lift it to A'. This clearly suggest that we should define our deformation functor to be smooth if the same lifting property applies to it. If at some point the lifting have to stop we say that the deformation functor is obstructed.
So let us see what parametrize the obstruction to our original deformation problem discussed in Section 3
It is clear that one can assume that the obstruction is taking place in the following way(after filtering an extension of Artin ring with one where all the kernels are one dimensional, so they behave as): I have lifted my first cohomology class $c \in F(k[\epsilon]) = H^1(X, T_X)$ up to an element $c' \in F(A)$ and now I am trying to lift it to A' via $f : A' \to A$ with $ker(f)$ 1 dimensional(and thus square 0) ideal.

Call $\tilde{X}$ an element of $F(A)$ lifting c. By the infinitesimal lifting property one gets an open cover of X, U_i with corresponding U_i' open cover of $\tilde{X}$ and trivializations $\varphi_i : U_i' \to U_i \times Spec(A)$, from wich you get a 1-cocycle of automorphisms over $Spec(A)$ $\theta_{i,j} \in Aut_{Spec(A)}(U_{i,j} \times Spec(A))$ wich you hope to lift to a 1-cocycle of automorphism $\theta'_{i,j} \in Aut_{Spec(A')}(U_{i,j} \times Spec(A'))$ wich restrict to $\theta_{i,j}$. So if you lift independently each $\theta_{i,j}$ to some $\theta'_{i,j}$ you find that $\theta'_{i,j}\theta'_{j,k}{\theta'_{i,k}}^{-1}$ is the identity when restricted to $Spec(A)$, thus it is giving you an element of $H^0(U_{i,j,k}, T_X)$ wich is easily checked to be in the kernel of the δ-map of the Cech complex giving you altogheter an element of $H^2(X, T_X)$, wich, as a cohomology class, does not depend on the open cover and on the choice of the lifting(they both change it by a coboundary). But on the other side to get a lifting we clearly want to be able to pick liftings of $\theta_{i,j}$ so that we get 0(that is also $\theta'_{i,j}$ is a 1-cocycle); that is, our lifting is possible if and only if this cohomology class is trivial. So we immediately conclude that if $H^2(X, T_X) = 0$ then there are no obstruction to the lifting and the deformation functor of X is smooth. In particular if X is a projective smooth curve, then the functor is unobstructed.

(We have just learned that starting with a curve and a cohomology class $c \in H^1(X, T_X)$ we can perform successive lifting of this to $k[x]/x^n$, using a projective embedding you see that this compatible family lifts to a curve over $k[[x]]$ wich lifts the first order deformations c you started with. In this sense you see how in this setting every tangent vector can be lifted to a family having him as first order approximation).

5 Number of parameters

We have seen in section 3 that the first order deformations of an algebraic variety X are naturally parametrized by $H^1(X, T_X)$, in section 4 we have seen that the obstructions to realize this first order deformation from "family"(in a quite formal sense) live naturally in $H^2(X, T_X)$, so in the particular case of a smooth projective curve we see that there are no obstruction. This tells us that our moduli problem is, formally, smooth, and we can go on computing dimension of tangent spaces: for a curve T_C is a line bundle an is the inverse of the canonical sheaf, thus by Serre duality you get that $dim_k(H^1(C, T_C)) = dim_k(H^0(C, 2K_C))$. Thus if $g(C) = 0$, then $2K_C$ has negative degree so there are no global section so we get 0 as number of parameters, if $g(C) = 1$ then the canonical divisor is trivial and we get 1 as number of parameters, if $g(C) \geq 2$ then $2K_C$ has degree $4g - 4$ so you get, by Riemann Roch, as number of parameters $4g - 4 - g + 1 = 3g - 3$.

References

[1]HartshorneDeformation theory [2]KodairaDeformation theory of complex structure