(1) Peter Bruin, Goss L-fctns

$\mathbb{F}_q$, char p; $C/\mathbb{F}_q$ sm. proj geom curve, $\infty \in C(\mathbb{F}_q)$, $A = \Gamma(\mathcal{O}_{C \setminus \infty})$, $A^\times = \mathbb{F}_q^\times$,

$K = \operatorname{Frac} A$, K_∞ = comp at ∞, $\mathbb{C}_\infty = \hat{\bar{K}}_\infty$

Stable curves All over base ring Λ, eg $\Lambda = \mathbb{Z}$ or $\mathbb{C}$ or ... excellent?

So far: - defined $\mathcal{M}_{g,n}$
- shown is alg. stack
- shown smooth $/\Lambda$

But not proper $/\Lambda$.
(def? $\mathcal{M}_{g,n} \to \Lambda$ not rep, so can't use idea in R's talk.)

Can talk about universal closedness for stacks, but then have to def the topology, which would take a while. So will use val. criterion as shortcut.

Def/Thm Let S a scheme & $\mathcal{X} \xrightarrow{F} S$ a DM stack. Assume S loc. noeth & F loc. fin. type. We say F is proper if:

(def ok bec same local for Et/sm. top, we can get)

- F is separated (ie $\Delta: \mathcal{X} \times_S \mathcal{X} \to \mathcal{X}$ proper (is rel. rep. by alg sp, so ok)),
- F is of fin type, (ie $\exists$ sm. cover by q.c.?)
- F satisfies the valuative criterion for ~~local~~ universal closedness (noeth):

$\forall$ ~~RMDVRs~~ DVRs R w frac. field k, $\forall$ 2-comm. diagrams

$$\begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow F \\ \operatorname{Spec} R & \longrightarrow & S \end{array},$$

$\exists$ k'/k field ext, & R' a val. ring of k' dominating R (def?), & a 2-comm diagram

$$\begin{array}{ccccc} \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \nearrow & \downarrow F \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R & \longrightarrow & S \end{array}$$

Moreover, may take K'/K fin. sep., & K complete & w alg closed res field!

Prop. $\mathcal{M}_{g,n}$ is <u>not</u> proper over Λ unless $g=0$ & $n=3$ (then $\mathcal{M}_{g,n} = \Lambda$!)

Pf. Let R a DVR, C/R curve of genus g, n marked pts, s.t.

- C/R has nodal sings, <u>not</u> smooth
- C_K/K smooth.
- C minimal regular

(Not enough just to write down a non-smooth curve over a DVR - 3 pt-good red'n)

Let R'/R ext. of DVRs (fin flat, local). Does $C_{K'}$ have a smooth model?

No. If it did, then the MRM would be smooth. But the MRM is obtained from $C \times_R R'$ by glueing in chain of d $\mathbb{P}^1$s at each node, where d = ram deg R'/R (see Liu).

□

Def: S scheme. A <u>nodal curve</u> over S is a proper, flat, fin pres $C \to S$ s.t. $\forall$ sep. cl. fields k, $\forall$ $p : \operatorname{Spec} k \to S$, have that $p^*C \to k$ has <u>'at worst nodal sings'</u>

i.e. '$\forall$ pts $x \in p^*C$, <u>either</u> $p^*C \to k$ is smooth at x, <u>or</u> have k-alg iso $\widehat{\mathcal{O}}_{p^*C, x} \cong \frac{k[[x,y]]}{xy}$.

---

marked pts on nodal curves must be smooth pts!

③ Want to compactify $M_{g,n}$ (eg. bcs Erik wants to do on the diagonal)

Have seen that smooth curves can degenerate to nodal curves. Maybe try taking the stack $\mathcal{M}_{g,n}$ of all nodal curves?

Prop: $\mathcal{M}_{g,n} \to \Lambda$ is not separated

Pf: Will show diag fails VCP. Idea: R a DVR, (R, m, K, k), $C/_R$ sm proper

$c \in C(k)$ (assume exists!), $\tilde{C} = Bl_c C$

Then ~~isom~~ $C_K \cong \tilde{C}_K$, but $C \not\cong \tilde{C}$, so

we get ~~an~~ a map $K \to Isom(C, \tilde{C})$ that does not extend to an R-map. □

(same pf shows taking 'all curves' not good)

So ~~we~~ too few smooth curves, too many nodal curves.

Def Let $C \overset{\leftarrow}{\to} S$ be nodal w n marked sm pts. ~~We say it is stable~~

Let $k = \bar{k}$, $s \in S(k)$. Let C ~~have~~ Y an irred compt C_s. A special pt on Y is a marked or non-sm pt.

We say C_s is stable if

- for every Y ~~of~~ w. $p_g(Y) = 0$, then Y has ≥ 3 special pts
- $p_g(Y) = 1$,

Say $C \overset{\leftarrow}{\to} S$ stable if it is so fibrewise.

Ex: A nodal curve $C/_{k, p_1 \ldots p_n}$ is stable iff $\#Aut(C_k, p_1, \ldots, p_n) < \infty$.

Write $\overline{M}_{g,n}$ for stack of stable curves.

Thm $\overline{M}_{g,n}$ is proper & smooth $/\Lambda$.

Pf Omitted. # VCP is ≡ to:

Let R a DVR, $C/_{k=\mathrm{Frac}\,R}$ a sm proper curve. Then $\exists\, \mathrm{Spec}\,R' \xrightarrow{f} \mathrm{Spec}\,R$ fin. flat map of DVRs & $C'/_R$ s.t.

$$f^*C \simeq C'_{k'} \quad (k' = \mathrm{Frac}\,R') \quad \left(\text{all w. marked pts}\right)$$

, $C'/_R$ is stable

'semistable red'n thm'. (□)

Have an open immersion $M_{g,n} \hookrightarrow \overline{M}_{g,n}$ unless $g=1$ & $n=0$ or $g=0$ & $n=0,1,2$.

'view smooth curve as stable curve'. exactly when $M_{g,n}$ not DM.

Q Why is it an open imm?

Let C/S stable. Wts $\exists\, U \hookrightarrow S$ open imm s.t. $\forall T \to S$, T factors uniquely via U iff C_T/T is smooth. Will actually do a bit more: will define $\mathrm{Sing}(C/S)$ (set of pts where C/S not smooth) in a functorial way.

1st Fitting ideal Let R a com. ring, $M \in R$-mod fin. pres. Let

$$\bigoplus_{m} R \xrightarrow{f} \bigoplus_{n} R \longrightarrow M \longrightarrow 0 \qquad \text{be a finite presentation}$$

Then f is given by an $n \times m$ matrix $(a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$

Then $\mathrm{Fitt}_1(M)$ is the ideal of R generated by the $(n-1)\times(n-1)$ minors of the matrix

Thm: $\mathrm{Fitt}_1(M)$ is indep of the pres., & if M is free of rk 1 then $\mathrm{Fitt}_1 M = 0$ (ex). Formation commutes w. étale B.C., completion etc.

Pf Omitted

Extend to (f. pres) sheaves of moduls in evident way.

⑤ **Def**: let C/S nodal. Define $\mathrm{Sing}(C/S)$ to be the closed subscheme of C/S corresp to $\mathrm{Fitt}_1(\Omega_{C/S})$

~~Prop: Formation~~

eg $S = \mathrm{Spec}\, k$, $C = \frac{k[x,y]}{xy} =: A$ (not nodal bc not proper but OK étale locally)

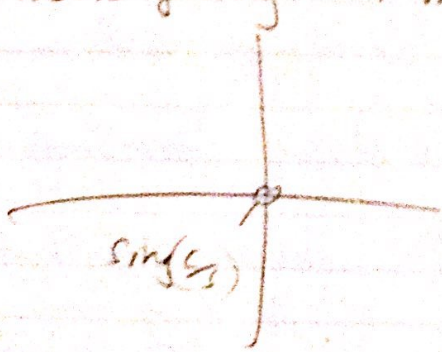
Then $\Omega^1_{C/S} = \frac{A\langle dx, dy\rangle}{(x\,dy - y\,dx)}$.

Presentation:

$$A \xrightarrow{(x,-y)} A \oplus A \to \Omega^1_{C/S} \to 0$$

So $\mathrm{Sing}(C/S)$ is cut out by the ideal gen by 1×1 minors of $(x,-y)$

ie $\mathrm{Sing}(C/S) = (x,y)$.



Prop: 1 - Formation of $\mathrm{Sing}(C/S)$ commutes w. arb. B.C over S

2 - $\mathrm{Sing}(C/S) = \phi \iff C/S$ smooth

3. - $\mathrm{Sing}(C/S)$ is finite unram over S

Pf 1. omitted

2. ~~smoot~~ - reduce to noeth base
- check smoothness on geom fibres
- apply (1)
- apply example above + étale loc (completion is OK, but needs care) w. él is formal

3. Clearly proper + q-finite + f. pres. Unram can be checked on geom fibres, then above eg. makes it clear

∎

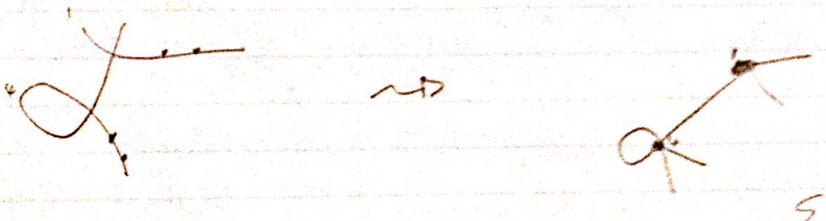
In particular, ~~above~~ (1) + (2) $\Rightarrow$ M_g is open in $\overline{M}_{g}$

Graphs of stable curves

$k = \bar{k}^{sep}$, C/S, $\sigma_1, \dots, \sigma_n \in C(S)$ nodal w. marked pts. Define the graph:

- vertices = irred comps of C
- ~~If u, v vertices then put one edge between them for each sing pt lying on~~
- for each sing pt, make an edge between the two irred comps containing it (maybe one or 2)
- can decorate each vertex w. a 'leg' or '$\frac{1}{2}$ edge' for each marking
- can label each vertex by its p_g.

eg

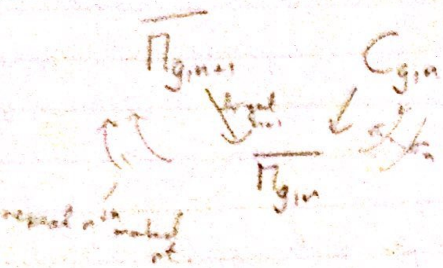


'Feynman diagrams'

Define genus of graph to be rk $H_1(\Gamma, \mathbb{Q})$ as top. space. Then

<u>Ex/Thm</u>: $p_a(C) = \sum_v p_g(v) + \text{genus}(\Gamma)$

(⑦ <u>The universal curve</u> Let • $2g-2+n>0$. Have

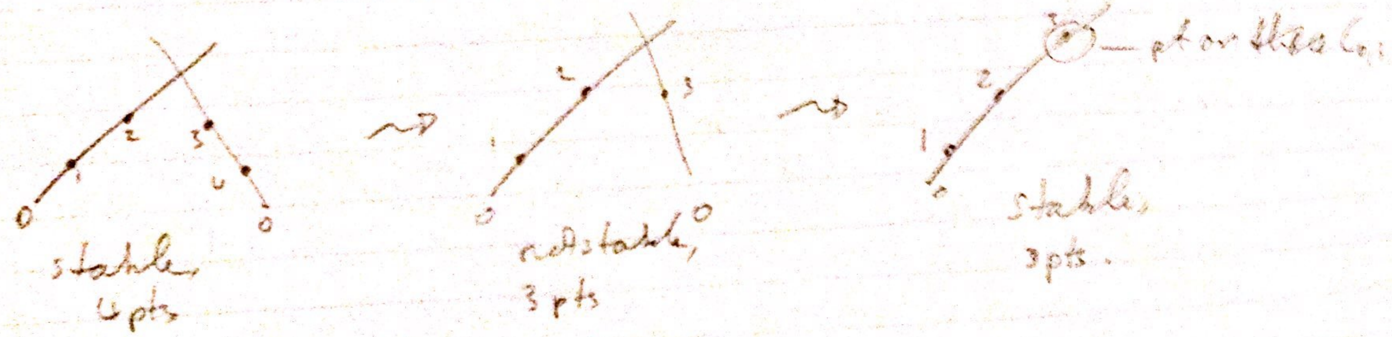


In fact, these are naturally isomorphic. To save time, just describe maps on pts. $C_{g,n}(k^{alg})$ is just a pt in $\overline{M}_{g,n}$ together w/ a pt on that curve.

Let $(C, \sigma_1, \ldots, \sigma_{n+1}) \in \overline{M}_{g,n+1}(k)$. Then $(C, \sigma_1, \ldots, \sigma_n)$ is nodal & has n marked pts, but may not be <u>stable</u>.

Answer: contract down the offending $\mathbb{P}^1$.

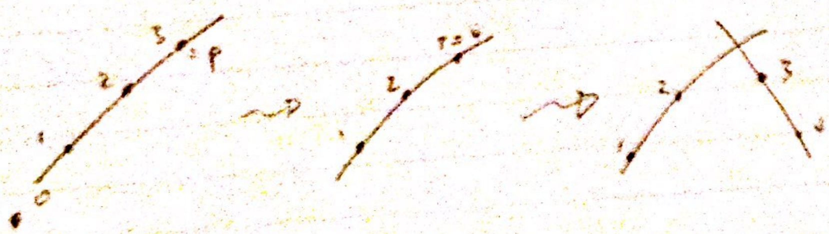
eg
$n=3$
$g=0$



Let $(C, \sigma_1, \ldots, \sigma_n, p) \in C_{g,n}(k)$. Setting $\sigma_{n+1} = p$ gives nodal n-pted curve, but pts may not be distinct if eg $p = \sigma_n$.

Answer: insert a $\mathbb{P}^1$

eg
$g=0$
$n=3$



Boundary: We know $M_{g,n} \hookrightarrow \overline{M}_{g,n}$ open imm.

Can we put a good red str. on boundary?

Yes. Working etale locally on $\overline{M}_{g,n}$, may assume

$$\mathrm{Sing}(C_{g,n}/\overline{M}_{g,n}) \to \overline{M}_{g,n} \text{ is a disj't union of closed immersions,}$$

say $\quad \| \quad Z_1, \dots, Z_r, \quad Z_i \hookrightarrow \overline{M}_{g,n}$

* Image of each Z_i is cut out by a reg. elt, say z_i. (needs pf!)

Define ~~reduced~~ boundary $\partial \overline{M}_{g,n}$ to be given by $\prod_{i=1}^{r} z_i$.

Formation commutes w. B C, gives that $\partial \overline{M}_{g,n}$ is NCD in $\overline{M}_{g,n}$.

More precise description

Fix a graph Γ of genus g w. n legs. Given a vertex v,

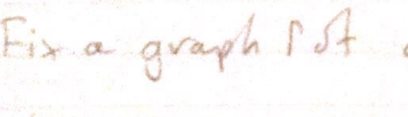
Let $E(v)$ = ends of edges in v,
$l(v)$ = legs on v.

Then make a map $\prod_v \overline{M}_{g(v), \#E(v)+\#l(v)} \longrightarrow \overline{M}_{g,n}$,

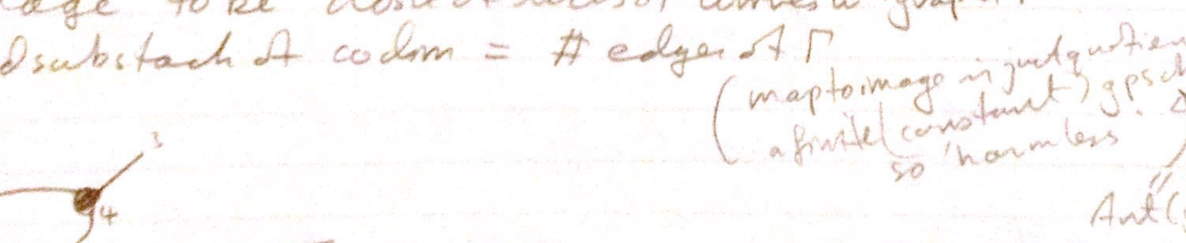
by gluing along endent sections (see eg. below).

Define image to be 'closed locus of curves w. graph Γ'
Is a closed substack of codim = # edges of Γ
(map to image is just quotient by a finite (constant) gp scheme, so 'harmless') ↑ = Aut(graph)

eg.



$$\overline{M}_{1,4} \times \overline{M}_{4,3} \longrightarrow \overline{M}_{6,3}$$



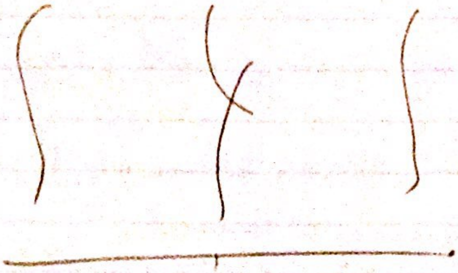
glue a to c (fibred coprod along)
glue b to d

(9) Warning: Can't just impose conditions fibrewise.

eg 'Let $F \subset \overline{\mathcal{M}}_{g,n}$ be substack of curves s.t. on every geom fibre, have ≥ 1 sing pts'

Might guess $F = \partial \overline{\mathcal{M}}_{g,n}$, but actually F not an alg stack.

Pf: Take a nodal EC / Spec $k[[t]]$, ~~closed~~ generically smooth



$$\begin{array}{c} C \\ \downarrow \\ S = \operatorname{Spec} k[[t]] \end{array}$$

$t=0$

Then look at the system

$$\frac{k[[t]]}{t} \leftarrow \frac{k[[t]]}{t^2} \leftarrow \frac{k[[t]]}{t^3} \leftarrow \cdots$$

Spec!! S_1 Spec!! S_2

$$S_1 \rightarrowtail S_2 \rightarrowtail S_3 \rightarrowtail \cdots$$

Let $C_i = C \times_S S_i$.

$\rightsquigarrow$

$$\begin{array}{ccccc} C_1 & \rightarrowtail & C_2 & \cdots \\ \downarrow & & \downarrow & \\ S_1 & \rightarrowtail & S_2 & \rightarrowtail & \cdots \end{array}$$

All C_i/S_i have exactly 1 sing pt on every fibre.

But the 'limiting object' C does not.

So we get maps $S_i \longrightarrow F$ $\forall i$, compatibly,
but no map $S \longrightarrow F$. Thus $\nexists$ rep of F.