Universiteit Leiden
Wiskunde en Natuurwetenschappen

Vak:
Naam:
Datum:
Studierichting:
Docent:
Collegekaartnummer:

Gromov - Witten invariants

Last time: gave formula for

$N_d := \#\{$ rational curves in $\mathbb{P}^2$ passing through $3d-1$ points in general position $\}$

For $d \geq 1$, $\overline{M}_{0,3d-1}(\mathbb{P}^2, d)$ = coarse moduli space of stable maps of degree d with $3d-1$ markings

For $i = 1, \dots, 3d-1$ have $\nu_i : \overline{M}_{0,3d-1}(\mathbb{P}^2, d) \to \mathbb{P}^2$, evaluation at i-th marking
Then $N_d = \#$ points in $\bigcap_{i=1}^{3d-1} \nu_i^{-1}(p_i)$, where $p_1, \dots, p_{3d-1} \in \mathbb{P}^2$ are pts in gen position

We generalize this to subvarieties $\Gamma_i \subset \mathbb{P}^r$, instead of points in $\mathbb{P}^2$.
We need some intersection theory. From now on we work over $k = \overline{k}$

Chow groups and rings.

Algebraic-geometric analogue of cohomology groups $H^*(X, \mathbb{Z})$ for a top space X.

From now on: X is an integral, noetherian scheme.

Def: A prime cycle of codimension $r \in \mathbb{Z}_{\geq 0}$, $V \subset X$, is a closed integral subscheme $V \hookrightarrow X$ of codimension r.

$Z^r(X) :=$ free abelian group generated by prime cycles $V \subseteq X$ of codim r.
example: $Z^1(X)$ = group of Weil divisors of X.

$Z(X) := \bigoplus_{r \geq 0} Z^r(X)$ abelian group.

Given a closed subscheme $Y \subset X$, can associate cycle $\langle Y \rangle \in Z(X)$ as follows: let $Y_1, \dots, Y_s$ be irred components (of Y_{red}). ~~Then~~ $\eta_1, \dots, \eta_s$ generic pts of Y corresponding to $Y_1, \dots, Y_s$. Then

$$\cancel{\langle Y \rangle = \sum \mathrm{length}(\mathcal{O}}$$

$$\langle Y \rangle = \sum_{i=1}^{s} \mathrm{length}(\mathcal{O}_{Y,\eta_i})\, Y_i .$$

<u>Define</u> $\mathrm{Rat}(X) \subset Z(X)$ subgroup generated by

$\langle F^{-1}(0) \rangle - \langle F^{-1}(\infty) \rangle \in Z(X)$ for all diagrams

$$\begin{tikzcd} Y \arrow[r, hook, "\text{cl. imm}"] \arrow[dr, "F"', "\text{flat}"'] & X \times \mathbb{P}^1 \arrow[d, "pr_1"] \\ & \mathbb{P}^1 \end{tikzcd}$$

<u>Def</u> $A(X) := Z(X) / \mathrm{Rat}(X)$ Chow group of X

Because of flatness condition in definition above, $\mathrm{Rat}(X)$ is homogeneous subgroup of $Z(X) = \oplus Z^k(X)$.

so

$$A(X) = \bigoplus_{k \geq 0} A^k(X) \quad \text{where } A^k(X) = Z^k(X) / \mathrm{Rat}(X) \cap Z^k(X)$$

For cl. subscheme $Y \subset X$, denote $[Y]$ its class in $A(X)$

Assume X is regular, q. projective over $k = \bar{k}$. Then there is product on $A(X)$. Given prime cycles $V, W \subset X$, we can change ~~these~~ the representatives of the classes $[V], [W] \in A(X)$ so that V and W meet properly (i.e. $\mathrm{codim}_X V + \mathrm{codim}_X W = \mathrm{codim}_X(V \cap W)$) and let

$$V \cdot W = \sum_{\substack{Z \subset X \text{ closed} \\ \text{integral} \\ Z \subset V \cap W}} i_Z(V,W)[Z], \quad \text{and extend by linearity}$$

where $i_Z(V,W) = \sum_{i \geq 0} (-1)^i \mathrm{length}_{\mathcal{O}_{X,Z}} \mathrm{Tor}_i^{\mathcal{O}_{X,Z}}\left(\frac{\mathcal{O}_{X,Z}}{I}, \frac{\mathcal{O}_{X,Z}}{J}\right)$ $\quad$ I, J ideals of V, W.

Because X is regular every $\mathcal{O}_{X,Z}$-module has finite Tor-dimension (TAG 0AZT), so the sum is finite.

Given $\alpha \in Z(X)$, $\beta \in \mathrm{Rat}(X)$, $\alpha \cdot \beta \in \mathrm{Rat}(X)$

Get <u>commutative graded ring</u> structure on $A(X) = \bigoplus_{k \geq 0} A^k(X)$

Functoriality: ~~f: X → Y proper of schemes~~

Def: $f: X \to Y$ proper of integral noetherian schemes, $V \subset X$ closed integral.
Define
$$f_* V \in Z(Y) \text{ by } \begin{cases} 0 & \text{if } \dim f(V) < \dim V \\ [k(V): k(f(V))]\,[f(V)] & \text{else} \end{cases}$$

Thm f_* induces group hom of graded groups $f_*: A(X) \to A(Y)$

Def $f: X \to Y$ flat of integral, noeth schemes, $V \subset Y$ cl. integral.
~~Define $f^* V \in Z(X)$ by $[f^{-1}(V)]$.~~
Define $f^* V := [f^{-1}(V)] \in A(X)$.

~~Rmk if X, Y regular~~

Thm f^* induces group hom $f^*: A(Y) \to A(X)$

Rmk if X and Y are regular, f^* is ring homomorphism

Projection formula: $f: X \to Y$ proper flat of regular schemes, $\alpha \in A(X)$, $\beta \in A(Y)$
then $f_*(\alpha \cdot f^* \beta) = (f_* \alpha) \cdot \beta$.

Example $A(\mathbb{P}^n_k) \cong \mathbb{Z}[x]/x^{n+1}$, H any hyperplane in $\mathbb{P}^n_k$.
$[H] \longleftarrow x$
So every prime cycle Γ of codim k is such that $[\Gamma] = n[H]^k$, $n \in \mathbb{Z}$.

Thm $A(\mathbb{P}^r \times \mathbb{P}^s) \cong A(\mathbb{P}^r) \otimes A(\mathbb{P}^s) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1})$

Class of diagonal $\Delta \hookrightarrow \mathbb{P}^r \times \mathbb{P}^r$, $[\Delta] \in A(\mathbb{P}^r \times \mathbb{P}^r)$

$[\Delta] = \sum_{i=0}^{r} c_i \alpha^i \beta^{r-i}$, $c_i \in \mathbb{Z}$. $\quad c_i = \deg([\Delta] \cdot \alpha^i \beta^{r-i})$

For $\Lambda \subset \mathbb{P}^r$, $\Gamma \subset \mathbb{P}^r$ of codim i and $r-i$, linear, $\Delta \cap (\Lambda \times \Gamma) = \Lambda \cap \Gamma$ is reduced point, so $c_i = 1$ $\forall i$, and
$$[\Delta] = \sum_{i=1}^{r} \alpha^i \beta^{r-i}$$

We work with moduli stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \to \mathrm{Spec}\,\bar{k}$

objects over base S are
$$\begin{array}{ccl} X & \longrightarrow & \mathbb{P}^r_S \\ \downarrow \varphi & \swarrow & \\ S & & \end{array}$$
with every geom fibre a stable map and $X \to S$ flat, proper + sections $S \to X$

$\overline{\mathcal{M}} := \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is D-M stack $\Rightarrow$ $\exists$ finite flat cover (of degree $\delta \in \mathbb{Z}_{\geq 1}$)

$Y \xrightarrow{F} \overline{\mathcal{M}}$ by a scheme.

If $f: X' \to X$ is finite flat deg δ of schemes, $f_*f^*: A(X) \to A(X)$ is multiplication by δ, hence $f_*f^*: A(X)\otimes_{\mathbb{Z}}\mathbb{Q} \to A(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ is iso

(*) Define $A(\overline{\mathcal{M}})_{\mathbb{Q}} := A(Y)\otimes_{\mathbb{Z}}\mathbb{Q}$.

~~A priori~~ Even though $\overline{\mathcal{M}}$ is smooth, Y may not be. A priori cannot intersect classes $[\nu_i^{-1}(\Gamma_i)]$ in $A(\overline{\mathcal{M}})_{\mathbb{Q}}$, $\Gamma_i \subset \mathbb{P}^r$. But $[\Gamma_i] = m_i[H]^{k_i} = m_i[\Gamma_i']$ where $k_i := \mathrm{codim}_{\mathbb{P}^r}\Gamma_i$, and $\Gamma_i' = \bigcap_{i=1}^{k} H_i$, H_i hyperplanes meeting transversally. Then Γ_i' is complete intersection, so has finite Tor-dim (Koszul complex). Since $Y \xrightarrow{F} \overline{\mathcal{M}} \xrightarrow{\nu_i} \mathbb{P}^r$ is flat, $F^{-1}\nu_i^{-1}(\Gamma_i')$ is complete intersection and can intersect with it.

~~So $A(\overline{\mathcal{M}})_{\mathbb{Q}}$ has structure of $A(\mathbb{P}^r)_{\mathbb{Q}}$-module (compatible with group homomorphism $A(\mathbb{P}^r)_{\mathbb{Q}} \to A(\mathcal{M})_{\mathbb{Q}}$~~

So the subgroup of $A(\overline{\mathcal{M}})_{\mathbb{Q}}$ spanned by $\{\nu_i^*\alpha \mid \alpha \in A(\mathbb{P}^r)_{\mathbb{Q}}, i=1,\dots,n\}$ has a ring structure.

(*) ERRATA: this is not a good definition. Indeed $f^*: A(X)_{\mathbb{Q}} \to A(X')_{\mathbb{Q}}$ in the example above is not an isomorphism, just an injection of rings. It is possible to develop a theory of intersection on stacks (Vistoli, "Intersection theory on algebraic stacks and their moduli spaces" for example). We don't really need to define $A(\overline{\mathcal{M}})_{\mathbb{Q}}$, just need to intersect classes that are pullbacks from $\mathbb{P}^r$.

Universiteit Leiden
Wiskunde en Natuurwetenschappen

Vak:
Naam:
Datum:
Studierichting:
Docent:
Collegekaartnummer:

Fix $d \geq 0$, $r \geq 2$, $k = \bar{k}$ field

Notation: $\overline{\mathcal{M}}_{(n,d)} = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r_k, d)$ $X := \mathbb{P}^r_k$, $\nu_i : \overline{\mathcal{M}}_{(n,d)} \to X$ evaluation maps

Prop: Let $U \subset \overline{\mathcal{M}}_{(n,d)}$ dense open. For generic choices of $\Gamma_1, \ldots, \Gamma_n \subset X$ closed integral, with $\sum \operatorname{codim} \Gamma_i = \dim \overline{\mathcal{M}}_{(n,d)}$, $\bigcap \nu_i^{-1}(\Gamma_i)$ consists of a finite number of reduced points.

we can choose $U = \mathcal{M}^* \subset \overline{\mathcal{M}}_{(n,d)}$ locus of automorphism-free stable maps with smooth source.

Lemma: for Γ_i as in the Prop,

$$\#\left(\bigcap \nu_i^{-1}(\Gamma_i)\right) = \int_{\overline{\mathcal{M}}} \prod_{i=1}^{n} (\nu_i^*([\Gamma_i]))$$

where $\int_{\overline{\mathcal{M}}} : A(\overline{\mathcal{M}}_{(n,d)})_{\mathbb{Q}} \longrightarrow \underset{A(\operatorname{spec} k)_{\mathbb{Q}}}{\overset{\shortparallel}{\mathbb{Q}}}$ is the pushforward map.

Pf follows from Prop, and $[\bigcap \nu_i^{-1}(\Gamma_i)] = \prod (\nu_i^*([\Gamma_i]))$, which is true because ν_i are flat.

Def The Gromov-Witten invariants of degree d associated to the classes $\gamma_1, \ldots, \gamma_n \in A(\mathbb{P}^r)_{\mathbb{Q}}$ is

$$I_d(\gamma_1, \ldots, \gamma_n) = \int_{\overline{\mathcal{M}}_{(n,d)}} \prod_{i=1}^{n} \nu_i^*(\gamma_i).$$

I_d is linear in its entries, and $I_d(\underline{\gamma})$ is 0 unless $\sum_{i=1}^{n} \operatorname{codim} \gamma_i = \dim \overline{\mathcal{M}}_{(n,$

Enumerative interpretation:

Prop : $\Gamma_1,\dots,\Gamma_n$ general irred subvar's of $\mathbb{P}^r$ of codim ≥ 2, with $\sum \operatorname{codim} \Gamma_i = \dim \overline{M}_{(n,d)}$.
Then $I_d([\Gamma_1],\dots,[\Gamma_n]) = \#$ rat'l curves of degree d incident to all Γ_i's.

Pf ~~$I_d([\Gamma_1],\dots,[\Gamma_n])$ counts # of stable n-pointed maps $(\mu, p_1,\dots,p_n)$ from $\mathbb{P}^1$ to $\mathbb{P}^r$ s.t. their markings are sent to the Γ_i. For gen choice of the Γ_i, and for every map μ in $\cap \nu_i^{-1}(\Gamma_i)$, we have $\mu^{-1}(\mu(p_i)) = \{p_i\}$ $\forall i = 1,\dots,n$~~

$I_d([\Gamma_1],\dots,[\Gamma_n])$ counts the number of n-pointed stable maps $(\mu : \mathbb{P}^1 \to \mathbb{P}^r, p_1,\dots,p_n)$ s.t. $\mu(p_i) \in \Gamma_i$ $\forall i = 1,\dots,n$. For general choice of the Γ_i and for every $\mu \in \cap \nu_i^{-1}(\Gamma_i)$, we have

$$\mu^{-1}(\mu(p_i)) = \{p_i\} \quad \forall i = 1,\dots,n.$$

Consequence: # rat'l curves of degree d incident to all Γ_i depends only on class of $[\Gamma_i]$ in the $\mathbb{Q}$-Chow ring. For example we can replace a conic hypersurface $\subset \mathbb{P}^r$ with ~~two hyperplan~~ the union of two hyperplanes.

Cor For $\mathbb{P}^2$, $I_d(\underbrace{h^2,\dots,h^2}_{3d-1}) = N_d$ (h is hyperplane class in $\mathbb{P}^2$)

Lemma: the only non-zero GW invariant with $d=0$ is for $n=3$ and $\sum \operatorname{codim} \gamma_i = r$. In this case,

$$I_0(\gamma_1,\gamma_2,\gamma_3) = \int_{\mathbb{P}^r} \gamma_1 \cdot \gamma_2 \cdot \gamma_3$$

Pf: $\overline{M}_{0,n}(\mathbb{P}^r, 0) = \overline{M}_{0,n} \times \mathbb{P}^r$ (empty for $n<3$). ν_i coincides with $pr_2 : \overline{M}_{0,n} \times \mathbb{P}^r \to \mathbb{P}^r$ for all $i = 1,\dots,n$

$$I_0(\gamma_1,\dots,\gamma_n) = \int_{\overline{M}_{(0,n)}} \nu_1^*\gamma_1 \cdot \ldots \cdot \nu_n^*\gamma_n = \int_{\overline{M}_{0,n}\times\mathbb{P}^r} pr_2^*(\gamma_1 \cdot \ldots \cdot \gamma_n) \cap [\overline{M}_{0,n} \times \mathbb{P}^r]$$

$$\overset{\text{projection formula}}{=} \int_{\mathbb{P}^r} \gamma_1 \cdot \ldots \cdot \gamma_n \cap \underbrace{pr_{2*}[\overline{M}_{0,n} \times \mathbb{P}^r]}_{\text{vanishes iff } \dim \overline{M}_{0,n} > 0 \text{ iff } n>3}.$$

□

__Lemma__: Only non-zero GW invariant with $n<3$ is
$$I_1(h^r,h^r)=1$$
__Pf__: ~~dxo~~ $r\geq 2$. can take $d>0$. $\dim \overline{M}_{(n,d)} = rd+r+d+n-3 \geq 2r+n-2$.
If $n<2$ $\sum \operatorname{codim}(\gamma_i) < \dim \overline{M}_{(n,d)}$. If $n=2$, only possibility $d=1$ and $\operatorname{codim} \gamma_i = r$. $\square$

There is commutative diagram

$$\begin{tikzcd}
\overline{M}_{0,n+1}(\mathbb{P}^r,d) \arrow[r, "\hat{\nu}_i"] \arrow[d, "\varepsilon"] & \mathbb{P}^r \\
\overline{M}_{0,n}(\mathbb{P}^r,d) \arrow[ur, "\nu_i"'] &
\end{tikzcd}$$

ε forgets last marking

So $\hat{\nu}_i^* = \varepsilon^* \nu_i^* : A(\mathbb{P}^r)_{\mathbb{Q}} \to A(\overline{M})_{\mathbb{Q}}$

__Lemma__: Only non-zero GW invar with $1 = h^0 \in A^0(\mathbb{P}^r)$ is
$$I_0(\gamma_1,\gamma_2,1)$$
__Pf__
Suppose $\gamma_{n+1}=1$. If $n\geq 3$ or $d>0$, have

$$\int \prod_{i=1}^{n} \hat{\nu}_i^*(\gamma_i)\cdot \hat{\nu}_{n+1}^*(1)\cdot [\overline{M}_{0,n+1}(\mathbb{P}^r,d)] = \int \prod_{i=1}^{n} \nu_i^*(\gamma_i)\cdot \underbrace{\varepsilon_*[\overline{M}_{0,n+1}(\mathbb{P}^r,d)]}_{=0}$$
$\square$

__Lemma__: Suppose $d>0$ and $\gamma_{n+1}=h$. Then
$$I_d(\gamma_1,\dots,\gamma_n,h) = I_d(\gamma_1,\dots,\gamma_n)\cdot d$$

__Pf__: let $H\subset \mathbb{P}^r$ be hyperplane. Then $\varepsilon| : \hat{\nu}_{n+1}^{-1}(H) \to \overline{M}_{0,n}(\mathbb{P}^r,d)$ is generically finite of degree d. Indeed a map $\mu \in \overline{M}_{0,n}(\mathbb{P}^r,d)$ has image intersecting H in d points, that can be chosen to be image of the $(n+1)$-st marking. So

$$\int_{\overline{M}_{(0,n+1)}} \prod_{i=1}^{n} \hat{\nu}_i^*(\gamma_i)\cdot \hat{\nu}_{n+1}^*(h) = \int_{\overline{M}_{0,n+1}} \prod_{i=1}^{n} \hat{\nu}_i^*(\gamma_i)\cdot [\hat{\nu}_{n+1}^{-1}(H)] =$$
$$= \int_{\overline{M}_{0,n}(\mathbb{P}^r,d)} \prod \nu_i^*(\gamma_i)\cdot \varepsilon_*[\hat{\nu}_{n+1}^{-1}(H)] = \int_{\overline{M}_{0,n}(\mathbb{P}^r,d)} \prod \nu_i^*(\gamma_i)\cdot d\cdot [\overline{M}_{0,n}(\mathbb{P}^r,d)]$$
$\square$

So, to compute GW invariants for $\mathbb{P}^2$ it's enough to know those including h^2, i.e. the numbers N_d.

Recursion

Recall that for disjoint sets A, B with $A \cup B = \{1, \dots, n\}$ and positive integers d_A, d_B with $d_A + d_B = d$ we have a boundary divisor

$$D := D(A, B, d_A, d_B) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$$

Let $\overline{\mathcal{M}}_A := \overline{\mathcal{M}}_{0, A \cup \{x\}}(\mathbb{P}^r, d_A)$ and $\overline{\mathcal{M}}_B := \overline{\mathcal{M}}_{0, B \cup \{x\}}(\mathbb{P}^r, d_B)$.

There is cartesian diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_A \times \overline{\mathcal{M}}_B \arrow[d, "\nu_{x_A} \times \nu_{x_B}"] \\ \mathbb{P}^r \arrow[r, hook, "\Delta"] & \mathbb{P}^r \times \mathbb{P}^r \end{tikzcd}$$

One uses the description of $[\Delta] \in A(\mathbb{P}^r \times \mathbb{P}^r)$ to get the following:

<u>Thm*</u>: $$\int_D \nu_1^*(\gamma_1) \cdot \ldots \cdot \nu_n^*(\gamma_n) = \sum_{e+f=r} I_{d_A}\left(\prod_{a \in A} \gamma_a \cdot h^e\right) \cdot I_{d_B}\left(\prod_{b \in B} \gamma_b \cdot h^f\right)$$

<u>Reconstruction theorem for $\mathbb{P}^r$</u> (Kontsevich-Manin-Ruan Tian)

All Gromov-Witten invariants for $\mathbb{P}^r$ can be computed recursively, the only necessary initial value is $I_1(h^r \cdot h^r) = 1$.

<u>Pf</u> (Sketch): to prove that the recursion terminates, need to express GW invar's in terms of GW invariants of lower degree of fewer marks. We can use the 4 lemmas before. Say we want to compute $I_d(\gamma_1, \dots, \gamma_n)$ in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Rearrange the γ_i so that γ_n has lowest codimension and write $\gamma_n = \lambda_1 \cap \lambda_2$, with λ_1, λ_2 of codimension smaller than γ_n. Now we consider $\overline{\mathcal{M}}_{0,n+1}(\mathbb{P}^r, d)$, denote marks by $m_1, m_2, p_1, \dots, p_{n-1}$. The class

$$\nu_{m_1}^*(\lambda_1) \cdot \nu_{m_2}^*(\lambda_2) \cdot \nu_{p_1}^*(\gamma_1) \cdot \ldots \cdot \nu_{p_n}^*(\gamma_n)$$

is the class of a curve. Intersect with the two linearly equivalent boundary divisor $D(m_1, m_2 | p_1, p_2) \sim D(m_1, p_1 | m_2, p_2)$ and integrate, applying the Thm *. We find a huge expression, where all G-W invariants with d_A and $d_B > 0$ are known, by induction. We only care about the contribution with $d_A = 0$ or $d_B = 0$. In this case, we have GW-invariants with λ_i in place of γ_n, so lower codimension: can apply recursion □