José: given an ell of form

relating Bleak & Toric formulae for ...

$$\begin{pmatrix} 1 & \alpha & 3 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \mapsto \Big(\underbrace{[\alpha]_{F^0}}_{=w}, \underbrace{[\gamma]_{F_0}}_{=z}, \underbrace{e^{2\pi i 3}}_{=s} \Big)$$

Quantum cohomology of $\mathbb{P}^n$

{ Recap on generating fctns.

Let R be a $\mathbb{Q}$-algebra (main eg: $R=\mathbb{Q}$). ~~Let $N_{i\geq 0}$ a seq. called~~ ~~the (exponential) generating~~

Let $r \in \mathbb{Z}_{>0}$.

Let $N : \mathbb{N}^r \longrightarrow R$. $\quad$ $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

~~Let $t_1 \dots t_r$ formal variables~~

Then the (exponential) gen fctn for N is

$$\sum_{a_1,\dots,a_r \in \mathbb{N}^r} \frac{x_1^{a_1} \cdots x_r^{a_r}}{a_1! \cdots a_r!} N(a_1,\dots,a_r) \in R[\![x_1,\dots,x_r]\!]$$

Basic case: $r=1$, $R=\mathbb{Q}$, so think of N as seq. of numbers, & gen fctn is

$$\sum_{a \in \mathbb{N}} \frac{x^a}{a!} N(a)$$

~~Because R is a $\mathbb{Q}$-algebra, we have der~~

We can formally differentiate power series, $\leadsto$

$$\frac{\partial}{\partial x_i} : R[\![x_1,\dots,x_r]\!] \longrightarrow R[\![x_1,\dots,x_r]\!]$$

<u>lemma</u>: $r=1$. Then $F_x := \frac{\partial F}{\partial x}$ is the gen fctn for the sequence

$$N'(a) = N(a+1)$$

<u>Pf</u>: $\frac{\partial}{\partial x}\Big(\sum_{a\geq 0} \frac{x^a}{a!} N(a)\Big) = \sum_{a\geq 1} \frac{x^{a-1}}{(a-1)!} N(a)$ $\square$

eg **Fibonacci #s** $r=1, R=\mathbb{Q}$ $N_0 = N_1 = 1, \quad N_{a+2} = N_{a+1} + N_a \quad \forall a \geq 0.$

Let F = gen fctn.

$\Rightarrow F_{xx} = F_x + F$ differential eq'n

$F(0) = 1, \quad F_x(0) = 1$ (initial condition)

(can solve diff. eq'n).

Products ~~Def~~ Again $x=1$. Then

$$\underbrace{\left(\sum_{a\geq 0} \frac{x^a}{a!} N(a)\right)}_{= F} \cdot \underbrace{\left(\sum_{b\geq 0} \frac{x^b}{b!} M(b)\right)}_{= G} = \sum_{c\geq 0} \frac{x^c}{c!} \left(\sum_{i=0}^{c} \binom{c}{i} N(i) M(c-i)\right)$$

So F.G is gen. fctn for sequence

$$\sum_{i=0}^{c} \binom{c}{i} N(i) M(c-i).$$

eg **Bell #s** $r=1, R=\mathbb{Q}$

$N(a) := \#$ ways to partition an a-elt set into disjoint nonempty subsets

We see $N(0) = 1, \quad N(1) = 1, \quad N(2) = 2.$

In gen, $N(a+1) = \sum_{i=0}^{a} \binom{a}{i} N(a-i)$.

(pf: let S have a elts, $S = \{p_1, \ldots, p_a\}$. ~~Then $S \cup \{p_0\}$~~
Then given a partition of $S \cup \{p_0\}$, classify according to # of the $\{p_1, \ldots, p_a\}$ in same partition as p_0.
Say i of them do.
There are $\binom{a}{i}$ choices of which ones, & are $N(a-i)$ ways to partition remainder.)

~~So $N(a+1)$~~

So $N(a+1) = \sum_{i=0}^{a} \binom{a}{i} N(a-i) \cdot 1$

Let F = gen fctn

$e^x = \sum_{i\geq 0} \frac{x^i}{i!}$ - gen fctn for 1,1,1, ...

Then $F_x = e^x F$

(

The G-W potential   fix $r \in \mathbb{Z}_{>0}$

Recall chow ring $A^*(\mathbb{P}^r) = \cancel{\mathbb{Z}}\, \mathbb{Q}[h] / (h^{r+1})$ h = class of hyperplane.

also $\mathbb{Q}$-coeffs today

Def: Given $\gamma_1, \ldots, \gamma_n \in A^*(\mathbb{P}^r)$, define

$$I(\gamma_1, \ldots, \gamma_n) = \sum_{d \geq 0} I_d(\gamma_1 \cdots \gamma_n)$$

$$\left(\text{recall} = \sum_{d \geq 0} \int_{\overline{M}_{0,n}(\mathbb{P}^r, d)} \nu_1^* \gamma_1 \cdots \nu_n^* \gamma_n \right)$$

Say each γ_i homog of deg c_i

Lemma: then $I(\gamma_1 \cdots \gamma_n) = I_{d_0}(\gamma_1, \ldots, \gamma_n)$

where $$d_0 = \frac{\sum_{i=1}^{n} c_i - r - n + 3}{r+1}$$ ~~$c_i = \text{codim}(\gamma_i)$~~.

Pf: $\text{codim}\left(\nu_1^* \gamma_1 \cdots \nu_n^* \gamma_n\right) = \sum_i c_i$

$\dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3$.

Set them equal $\Rightarrow$ done.

$\square$

The GW-potential ... $\Phi \in A(\mathbb{P}^r)[[x_0, \dots, x_r]]$,

defined by

$$\phi = \sum_{a_0, \dots, a_r \geq 0} \frac{x_0^{a_0} \cdots x_r^{a_r}}{a_0! \cdots a_r!} I\left((h^0)^{\cdot a_0} \cdot (h^1)^{\cdot a_1} \cdots (h^r)^{\cdot a_r}\right)$$

take care: h^r is r^{th} power of h in $A^*(\mathbb{P}^r)$,

so $h^{r+1} = 0$.

But $I((h^0)^{\cdot a_0} \cdots (h^r)^{\cdot a_r})$ means $I(\underbrace{h^0 \cdot h^0 \cdots h^0}_{a_0 \text{ times}} \cdot h^1 \cdots \cdots h^r)$

$$\| $$

$n = a_0 + a_1 + \dots + a_r$ $\quad \displaystyle\sum_{d \geq 0} \int_{\overline{M}_{0,n}(\mathbb{P}^r, d)} ev_1^* h^0 \, ev_2^* h^0 \cdots \cdots ev_n^* h^r$

<u>Compact notation</u>: $\underline{a} = (a_0, \dots, a_r) \in \mathbb{N}^{r+1}$

Then set $\underline{x}^{\underline{a}} = x_0^{a_0} \cdots x_r^{a_r}$, $\quad \underline{a}! = a_0! \cdots a_r!$,

$$\underline{h}^{\underline{a}} = (h^0)^{\cdot a_0} \cdots (h^r)^{\cdot a_r}$$

Then $$\phi = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I(\underline{h}^{\underline{a}}).$$

~~Formally~~ differentiating the G-W potential

Let $0 \le i \le r$. Then

$$\Phi_i := \frac{\partial \Phi}{\partial x_i} = \sum_{a_0 \ldots a_r \ge 0} \frac{x_0^{a_0} \cdots x_i^{a_i} \cdots x_r^{a_r}}{a_0! \cdots (a_i - 1)! \cdots a_r!} I(\ldots)$$

$$= \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I(\underline{h}^{\underline{a}} \cdot h^i)$$

Sim,

$$\Phi_{ijk} = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I(\underline{h}^{\underline{a}} h^i h^j h^k)$$

Def: As a $\mathbb{Q}$-module, the Quantum cohomology ring

$QA := A^*(\mathbb{P}^r)[[x_0, \ldots, x_r]] = A^*(\mathbb{P}^r) \otimes_{\mathbb{Q}} \mathbb{Q}[[x_0, \ldots, x_r]]$

as $A^*(\mathbb{P}^r)$ fin as $\mathbb{Q}$-module.

Has natural $\mathbb{Q}[[x_0, \ldots, x_r]]$-module structure.

Want to make QA into a ring. A basis as $\mathbb{Q}[[x_0, \ldots, x_r]]$ module is given by $h^0, \ldots, h^r$, so enough to say how to multiply these.

Def

$$h^i * h^j = \sum_{\substack{e+f=r \\ e, f \ge 0}} \Phi_{ije} h^f$$

OK, that's it: we have defined a ~~map~~ $\mathbb{Q}[[x_0, \ldots, x_r]]$-module map $*: QA \otimes QA \to QA$.

<u>Lem</u>: $*$ is commutative

$h^0 * - = id = - * h^0$

<u>Pf</u> obvious. □

Later: $*$ is associative (here we ~~…~~ see recursion again!)

<u>Sanity check</u>

First, note that if we project ~~restrict~~ to $\deg_x = 0$ we get a map

$$A^*(\mathbb{P}^r) \times A^*(\mathbb{P}^r) \longrightarrow A^*(\mathbb{P}^r).$$

What is it?

<u>claim</u>: $$h^i * h^j\big|_{\deg_x 0} = \begin{cases} h^{i+j} & i+j \le r \\ h^{i+j-r-1} & i+j \ge r+1 \\ \cancel{0} & \cancel{i+j \ge r+\ldots} \end{cases}$$ remember $0 \le i,j \le r$, $h^{r+1} = 0$

one bit missing

(<u>Pf</u>) (omit from talk!) Well, $h^i * h^j\big|_{\deg_x 0} = \sum_{0 \le e \le r} T_e \ldots$

~~Case 1 $i+j \le r$~~

$$= \sum_{0 \le e \le r} h^{r-e} \sum_{d \ge 0} I_d(h^i h^j h^e)$$

Now $I_d(h^i h^j h^e) = 0$ unless $d = \frac{i+j+e-r}{r+1}$.

So $$h^i * h^j\big|_{\deg 0} = \sum_{0 \le e \le r} h^{r-e} \begin{cases} I_{\frac{i+j+e-r}{r+1}}(h^i h^j h^e) & \text{if } \in \mathbb{Z} \\ 0 & \text{else.} \end{cases}$$

Case 1. $i+j \le r$. Then $-r \le i+j+e-r \le r$. So if it is in $(r+1)\mathbb{Z}$ then $i+j+e-r = 0$, so $e = r-i-j$, $r-e = i+j$

Then $$h^i * h^j\big|_{\deg 0} = h^{i+j} \underbrace{I_0(h^i h^j h^{r-i-j})}_{= \int_{\mathbb{P}^r} h^i \cdot h^j \cdot h^{r-i-j}} = h^{i+j}$$

<u>Case 2</u>: $i+j=r+1$. Then $1 \leq i+j+e-r \leq r+1$, so if $r+1 \mid i+j+e-r$ then $i+j+e-r = r+1$, so $\underline{e=r}$.

Then $$h^i * h^j \big|_{\deg 0} = h^0\, I_1(h^i \cdot h^j \cdot h^r)$$
$$= h^0$$

<u>Case 3</u>: $i+j \geq r+2$ ($2r \geq$). Then $\cancel{}\,2 \leq i+j+e-r \leq 2r$. So if $r+1 \mid i+j+e-r$ then $i+j+e-r = r+1$ as before, so $r-e = i+j-r-1$; $e = 2r+1-i-j < r$

$$h^i * h^j \big|_{\deg 0} = h^{i+j-r-1}\, \underbrace{I_1(h^i \cdot h^j \cdot h^{2r+1-i-j})}_{1}$$
$$= h^{i+j-r-1}$$

1 – why? ~~Linear span of~~ (~~span of line~~ line that goes through) h^i & h^j hasd ~~hasdim~~. TBH, I just checked a few egs.

<u>TO DO!</u>

Associativity

<u>Thm</u> The map $*$ is associative, ie

$$(\dagger) \quad (h^i * h^j) * h^k = h^i * (h^j * h^k) \qquad \forall i,j,k$$

<u>Pf</u> (outline):

$$\mathrm{LHS}(\dagger) = \sum_{e+f=r} \sum_{\ell+m=r} \phi_{ije}\,\phi_{fk\ell}\,h^m$$

$$\mathrm{RHS}(\dagger) = \sum_{e+f=r} \sum_{\ell+m=r} \phi_{jke}\,\phi_{fi\ell}\,h^m.$$

Now the h^m are lin indep, so $(\dagger)$ is $\equiv$ to

$$\forall i,j,k,l. \quad \sum_{e+f=r} \phi_{ije}\,\phi_{fk\ell} = \sum_{e+f=r} \phi_{jke}\,\phi_{fi\ell} \qquad \text{'WDVV differential eq'ns'}$$

$(\heartsuit)$

Now by product rule for gen fctns, have that LHS$(\heartsuit)$ is gen fctn for

$$\sum_{e+f=r} \sum_{\underline{n}_A+\underline{n}_B=\underline{n}} \binom{\underline{n}}{\underline{n}_A} I(\underline{h}^{\underline{n}_A}.h^i.h^j.h^e)\, I(\underline{h}^{\underline{n}_B}h^f.h^k h^\ell)$$

ie $\frac{\underline{n}!}{\underline{n}_A!\,\underline{n}_B!}$

& sim for RHS$(\heartsuit)$.

Equating ~~degree~~ coeffs of monomials in the $\underline{x}$, the WDVV are $\equiv$ to

$$\sum_{e+f=r} \sum_{\underline{n}_A+\underline{n}_B=\underline{n}} \binom{\underline{n}}{\underline{n}_A} I(\underline{h}^{\underline{n}_A}h^i h^j h^e)\, I(\underline{h}^{\underline{n}_B} h^f h^k h^\ell)$$

$$= \sum_{e+f=r} \sum_{\underline{n}_A+\underline{n}_B=\underline{n}} \binom{\underline{n}}{\underline{n}_A} I(\underline{h}^{\underline{n}_A} h^j h^k h^e)\, I(\underline{h}^{\underline{n}_B} h^f h^i h^\ell)$$

This $=$ comes from $D(p_1 p_2 \mid p_3 p_4) \equiv D(p_2 p_3 \mid p_1 p_4)$

in v. similar way to in Erik's Gubio talks - details omitted.

Uses splitting lemma.

The classical & quantum potentials

Recall $\phi = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} \underbrace{\sum_{d \geq 0} I_d(\underline{h}^{\underline{a}})}_{I(\underline{h}^{\underline{a}})}$.

Define $I_+(\underline{h}^{\underline{a}}) = \sum_{d>0} I_d(\underline{h}^{\underline{a}})$.

Let $\underset{\text{classical potential}}{\phi^{cl}} = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I_0(\underline{h}^{\underline{a}})$ & $\underset{\text{+ve-deg part.}}{\Gamma} = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I_+(\underline{h}^{\underline{a}})$

$\left[\right.$ Recall $I_0(\gamma_1 \cdots \gamma_n) = 0$ unless $n=3$ & $\sum_i \text{codim}\, \gamma_i = r$, in which case $I_0(\gamma_1 \cdot \gamma_2 \cdot \gamma_3) = \int_{\mathbb{P}^2} \gamma_1 \gamma_2 \gamma_3$

So $\phi^{cl}_{ijk} = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I_0(\underline{h}^{\underline{a}} h^i h^j h^k)$

$= I_0(h^i h^j h^k) = h^{i+j+k}$

So have $h^i h^j = \sum_{e+f=r} \phi^{cl}_{ije} h^f$

'the third derivatives of classical potential are structure constants for classical product

In gen, $h^i * h^j = h^{i+j} + \sum_{e+f=r} \Gamma_{ije} h^f$.

Special case of $\mathbb{P}^2$

3 classes: h^0, h^1, h^2.

Note $\Gamma_{ijk} = 0$ if ≥ 1 of $i,j,k = 0$.
(cf. Giulio's lecture)

We find

$$h^1 * h^1 = h^2 + \Gamma_{111} h^1 + \Gamma_{112} h^0$$

$$h^1 * h^2 = \Gamma_{121} h^1 + \Gamma_{122} h^0$$

$$h^2 * h^2 = \Gamma_{221} h^1 + \Gamma_{222} h^0.$$

Associativity ~~[illegible]~~ tells us

(only 1 other non-triv. case)

$$(\dagger) \quad (h^1 * h^1) * h^2 = h^1 * (h^1 * h^2).$$

$$LHS = \Gamma_{221} h^1 + \Gamma_{222} h^0 + \Gamma_{111} (\Gamma_{121} h^1 + \Gamma_{122} h^0) + \Gamma_{112} h^0$$

$$RHS = \Gamma_{121} (h^2 + \Gamma_{111} h^1 + \Gamma_{112} h^0) + \Gamma_{122} h^1$$

Equate coeffs of $h^0 \to \Gamma_{222} + \Gamma_{111}\Gamma_{122} = \Gamma_{112}\Gamma_{112}$.

(equation in $A^*(\mathbb{P}^2)[[x_0, x_1, x_2]]$)

Set $x_0 = x_1 = 0$; $\Gamma_{ijk}(x_0 = x_1 = 0) = \sum_{a \geq 0} \frac{x_2^a}{a!} I_+((h^2)^a h^i h^j h^k)$

So applying this + product rule to $(\dagger)$, we obtain

$$I_+((h^2)^{\cdot n} h^2 h^2 h^2) + \sum_{n_A + n_B = n} \binom{n}{n_A} I_+((h^2)^{\cdot n_A} h^1 h^1 h^1) I_+((h^2)^{\cdot n_B} h^1 h^2 h^2)$$

$$= \sum_{n_A + n_B = n} \binom{n}{n_A} I_+((h^2)^{\cdot n_A} h^1 h^1 h^2) I_+((h^2)^{\cdot n_B} h^1 h^1 h^2)$$

Keeping track of dimensions & ~~[illegible]~~ using Giulio's rule for ~~[illegible]~~ deleting hyperplanes, get eg

$$I_+((h^2)^{\cdot n_A} h^1 h^1 h^2) = d_A^2 N_{d_A}.$$

N_{d_A} = # deg d_A curves through $3d_A - 1$ pts in $\mathbb{P}^2$!

Apply to each of the others, get

$$N_d + \sum_{d_A+d_B=d} \binom{3d-4}{3d_A-1} d_A^3 N_{d_A} d_B N_{d_B}$$

$$= \sum_{d_A+d_B=d} \binom{3d-4}{3d_A-2} d_A^2 N_{d_A} d_B^2 N_{d_B}.$$

- cf. Erik's lecture, recursion for the N_d.

Small Q coh ring:

Traditional version: set all $x_i = 0$ except x_1.

then find

$$\Phi_{ijk}(x_1) = \sum_{a\geq 0} \frac{x_1^a}{a!} \sum_{d\geq 0} d^a I_d(h^i \cdot h^j \cdot h^k).$$

Set $q = \exp(x_1)$, then find

$$\Phi_{ijk}(x_1) = I_0(h^i h^j \cdot h^k) + q I_1(h^i h^j h^k) \quad \text{so}$$

$$h^i *_{small} h^j = \begin{cases} h^{i+j} & i+j \leq r \\ q h^{i+j-r-1} & i+j > r \end{cases}$$

So small q-coh ring of $\mathbb{P}^r$ is

$$\frac{\mathbb{Z}[h,q]}{(h^{r+1}-q)}.$$