The Grothendieck group of GL_n

Raymond van Bommel, rbommel@math.leidenuniv.nl
Supervisor: Joël Riou

The idea of this project is to give an alternative and easier proof to theorem 4 of [2, p. 49] in the case where $G = \mathrm{GL}_n$ and $k \subset \mathbb{C}$ a field.

1 Semisimplicity

In this section we want to prove the semisimplicity of GL_n-modules over k. Let $0 \to \rho' \to \rho \to \rho'' \to 0$ be an exact sequence of representations over the algebraic group $\mathrm{GL}_{n,k}$ over a field k of characteristic 0. Our goal can be reformulated as follows.

Theorem 1. *The functor* $\mathrm{Hom}(\rho'', -)$ *from the category of representations of* GL_n *over* k *to the category of* k*-modules is exact.*

Actually, this theorem would yield that

$$0 \to \mathrm{Hom}(\rho'', \rho') \to \mathrm{Hom}(\rho'', \rho) \to \mathrm{Hom}(\rho'', \rho'') \to 0$$

is exact and hence that the identity $\rho'' \to \rho''$ is the image of an element of $s \in \mathrm{Hom}(\rho'', \rho)$. This element is a section of the original exact sequence.

1.1 Compatibility with extension of scalars

In this section we will prove that theorem 1 is compatible with extension of scalars in the following sense.

Lemma 2. *Let* $k_1 \subset k_2$ *be two fields of characteristic 0. Then theorem 1 holds for* $k = k_1$ *if it holds for* $k = k_2$*.*

Proof. There is a natural map $Z : \mathrm{Hom}_{\mathrm{GL}_n,k_1}(V, W) \otimes k_2 \to \mathrm{Hom}_{\mathrm{GL}_n,k_2}(V \otimes k_2, W \otimes k_2)$ and this map is injective as $\mathrm{Hom}_{\mathrm{GL}_n,k_i}(V, W) \subset \mathrm{Hom}_{k_i}(V, W)$ (for $i = 1, 2$) and Z is the restriction of the isomorphism $\mathrm{Hom}_{k_1}(V, W) \otimes k_2 \cong \mathrm{Hom}_{k_2}(V \otimes k_2, W \otimes k_2)$. Next we will proof that Z is surjective.

Let $\phi \in \mathrm{Hom}_{\mathrm{GL}_n,k_2}(V \otimes k_2, W \otimes k_2)$. We can consider ϕ as a matrix and let $S \subset k_2$ be the k_1-vector space the matrix' coefficients generate. It is finite dimensional. Let $e_1, \ldots, e_j$ be a basis. As the action of $\mathrm{GL}_n, k_1 \subset \mathrm{GL}_n, k_2$ acts k_1-linear, the $k_1 e_i$-component ϕ_i of the map $\phi|_V$ is a morphism of GL_n, k_1-modules. Furthermore, ϕ_i is of the form $Z(\psi_i \otimes e_i)$ where $\psi_i \in \mathrm{Hom}_{\mathrm{GL}_n, k_1}(V, W)$. As $\phi = Z(\sum_i \psi_i \otimes e_i)$, we have proven the surjectivity now.

As $- \otimes k_2$ is an exact functor the statement immediately follows. □

Remark 3. To prove the statement for fields of characteristic 0 not contained in $\mathbb{C}$ we notice that a statement like this lemma holds for inductive limits and that every field of characteristic 0 is an inductive limit of subfields of $\mathbb{C}$.

1.2 Proof for $k = \mathbb{C}$

A representation ρ of GL_n over $k = \mathbb{C}$ induces a representation V of the group $\mathrm{GL}_n(\mathbb{C})$ where $\mathrm{GL}_n(\mathbb{C})$ has the usual topology. As $\mathrm{GL}_n \to \mathrm{Aut}_V$ is a morphism of varieties, the induced representation is smooth. We restrict this representation to the group $U_n \subset \mathrm{GL}_n$ of unitary matrices, call it V. In the same way ρ' and ρ'' induce representations V' and V'' of U_n. Now we will use the following fact to proof that the sequence $0 \to V' \to V \to V'' \to 0$ splits.

Fact 4. *Every locally compact Hausdorff topological group has a Haar measure.*

As U_n is a locally compact Hausdorff topological group we can and will equip it with a Haar measure and as U_n is abelian, this measure will be both right- and left-invariant. Furthermore we may and do suppose that the measure of the whole group U_n is 1 as U_n is compact.

Equip V with an arbitrary inner product $\langle \cdot, \cdot \rangle$. Then consider the map

$$B : V \times V \to \mathbb{C} : (v_1, v_2) \mapsto \int_{U_n} \langle g v_1, g v_2 \rangle dg.$$

Proposition 5. *The map B is an inner product of V that is U_n-invariant.*

Proof. Notice that $B(v, v) = \int \langle gv, gv \rangle dg$ is the integral of a non-negative function and hence it is non-negative. We also deduce immediately that $B(v, v) = 0$ if and only if $v = 0$. Furthermore, B is clearly linear in the first argument as $\langle \cdot, \cdot \rangle$ is linear in the first argument and in the same way we have $B(v_2, v_1) = \overline{B(v_1, v_2)}$. Hence, B is an inner product.

Furthermore,

$$B(v_1, v_2) = \int \langle gv_1, gv_2 \rangle dg = \int \langle gg_3v_1, gg_3v_2 \rangle dg = B(g_3v_1, g_3v_2),$$

as the Haar measure is U_n-invariant. □

Let W be a space orthogonal to V' in V with respect to the inner product B. Then for all $g \in U_n$, $w \in W$ and $v \in V'$ we have $B(gw, v) = B(w, g^{-1}v) = 0$ as $g^{-1}v \in V'$ and $w \in W$. Hence we have $gw \in W$ and we deduce that W is not only a subspace but in fact a subrepresentation ρ_W of V.

This yields an exact sequence of representations $0 \to V' \to V \to W \to 0$. In particular, W is isomorphic to V''. Finally, because $W \subset V$, this gives us a way to split the exact sequence as we wanted to do.

The subspace W induces a subspace of ρ complement to ρ' and isomorphic to ρ''. Hence ρ'' is fixed by the subgroup $U_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$. By proposition 12.1 of [1, p. 130] the stabilizer of ρ'' is a (Zariski closed) subgroup of GL_n. The following theorem will prove that ρ'' is in fact GL_n-invariant and concludes the proof that the exact sequence splits.

Lemma 6. *The subset $U_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_{n,\mathbb{C}}$ is Zariski dense.*

Proof. We will prove that $U_n(\mathbb{C})$ is dense in $\mathrm{GL}_n(\mathbb{C})$ which is dense in $\mathrm{GL}_{n,\mathbb{C}}$.

Let f be a polynomial on $\mathrm{GL}_n(\mathbb{C})$ that is zero on $U_n(\mathbb{C})$. We will prove that f is the zero polynomial. Consider the map $\exp : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ that exponentiates a matrix. It is known to be a surjective analytic function. In particular the function $g = f \circ \exp$ is analytic. We will prove that it is the zero function, which by the surjective of exp also proves that $f = 0$.

Suppose that $M \in \mathrm{Mat}_n(\mathbb{C})$ is such that $M = -M^*$. Then

$$(\exp M)^* = \sum_{n=0}^{\infty} \frac{1}{n!}(M^*)^n = \sum_{n=0}^{\infty} \frac{1}{n}(-M)^n = \exp(-M) = \exp(M)^{-1}.$$

Hence, $\exp(M) \in U_n(\mathbb{C})$ and $g(M) = 0$ for all $M \in \mathrm{Mat}_n(\mathbb{C})$ such that $M = -M^*$. For $i, j \in \{1, \ldots, n\}$ let E_{ij} be the matrix with a 1 in the (i, j)-th entry and zeros elsewhere. For $i \in \{1, \ldots, n\}$ let $A_i = i \cdot E_{ii}$. For $1 \leqslant i < j \leqslant n$ let $B_{ij} = E_{ij} - E_{ji}$ and let $C_{ij} = iE_{ij} + iE_{ji}$. Then the A_i, B_{ij} and C_{ij} together form a $\mathbb{C}$-basis of the vector space $\mathrm{Mat}_n(\mathbb{C})$. Moreover the basis vectors satisfy $M = -M^*$.

In other words, we can identify $\mathrm{Mat}_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$ in such a way that in this identification we have $g|_{\mathbb{R}^{n^2}} = 0$. By the theory of complex analysis now follows that all partial derivatives of g in the point $0 \in \mathbb{C}^{n^2}$ must be 0 and as g is analytic this yields that $g = 0$ on the whole $\mathbb{C}^{n^2} \cong \mathrm{Mat}_n(\mathbb{C})$. $\square$

2 Character theory

Let ρ be a representation of GL_n over k. Then ρ induces a representation of $\mathrm{GL}_n(k)$ and we define its character to be the function $\chi_\rho : \mathrm{GL}_n(k) \to k : g \mapsto \mathrm{Tr}(\rho(g))$. The goal of this chapter is to prove the following theorem.

Theorem 7. *Two finite-dimensional representations ρ_1 and ρ_2 of GL_n over k are isomorphic if and only if their characters are equal.*

2.1 Preliminary results

The following results will be needed to prove theorem 7.

Lemma 8. *Let A be a ring and E be a simple A-module. Let N be the Jacobsen radical of A. Then $NE = 0$.*

Proof. As E is simple it is generated by one element, say $e \in E$. Let $I = \mathrm{Ann}(e) := \{a \in A : ae = 0\}$; it is a left ideal of A. Then $E \cong A/I$. The submodules of E correspond to the left ideals of the ring A containing I. As E is simple, we decude that there are two such ideals and hence that I is a maximal left ideal of A. But then we have $I \supset N$ and hence $NE = 0$. □

Corollary 9. *Let A be a ring and E be a semisimple A-module. Let N be the Jacobsen radical of A. Then $NE = 0$.*

Theorem 10 (Artin-Wedderburn)**.** *Let A be a commutative ring. Suppose that A is artinian and that its Jacobsen radical is zero. Then A is a finite product of matrix rings over division rings.* □

2.2 Proof of the theorem

Proof. Let V_1 and V_2 be two $k[\mathrm{GL}_n(k)]$-modules that are finite-dimensional as k-vector space and have the same character. By the results from the first chapter, we know that V_1 and V_2 are semisimple. Let N be the kernel of the natural map $\mathrm{GL}_n(k) \to \mathrm{End}_k(V_1 \oplus V_2)$ and let $B = \mathrm{GL}_n(k)/N$. Then V_1 and V_2 are B-modules and B acts faithfully on $V_1 \oplus V_2$, hence B is finite dimensional as k-vector space.

Of course V_1 and V_2 are semisimple B-modules, as their simple components remain simple over B. Hence, by corollary 9 the Jacobsen radical of B acts trivially on both V_1 and V_2 and hence it is 0. As B is finite dimensional over k it is certainly artinian. Hence by the Artin-Weddernburn theorem, we have

$$B = \mathrm{Mat}(D_1, n_1) \times \cdots \times \mathrm{Mat}(D_s, n_s),$$

where for $i = 1, \ldots, s$ we have $n_i \in \mathbb{Z}_{>0}$ and D_i is a finite dimensional division algebra over k.

Notice that as a B-module $\mathrm{Mat}(D_i, n_i)$ is isomorphic to the product of n_i copies of the simple module $D_i^{n_i}$, where B acts in the obvious way (the i-th factor acts by multiplication and the other factors by zero). In particular we deduce that the simple modules are isomorphic to the $D_i^{n_i}$.

Let $\pi_i \in B$ be such that $\pi_i|_{\mathrm{Mat}(D_i,n_i)} = 1$ and $\pi_i|_{\mathrm{Mat}(D_j,n_j)} = 0$ for all $i, j \in \{1, \ldots, s\}$ such that $i \neq j$. Then $\chi_{V_1}(\pi_i)$ is the number of factors $D_i^{n_i}$ in V_1. The same holds for V_2. Together with the fact that the characteristic of k is 0 this proves that V_1 and V_2 are isomorphic.

For two $\mathrm{GL}_{n,k}$-modules with the same character, their underlying $\mathrm{GL}_n(k)$-modules are isomorphic. This gives a isomorphism of vector spaces $V_1 \to V_2$ that commutes with the action of $\mathrm{GL}_n(k)$. As $\mathrm{GL}_n(k) \subset \mathrm{GL}_{n,k}$ is Zariski dense, the isomorphism in facts commutes with $\mathrm{GL}_{n,k}$ and is a isomorphism of $\mathrm{GL}_{n,k}$-modules. □

3 Main statement

In the last chapter we will finally proof theorem 4 of [2, p. 49].

By the corollary of proposition 7 of [2, p. 48] the Grothendieck group of the subgroup $D \subset \mathrm{GL}_n$ of diagonal matrices is isomorphic to the group $H := \mathbb{Z}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$. This isomorphism is called $\mathrm{ch} : \mathrm{R}_k(D) \to H$. If V is a D-comodule, then the $X_1^{i_1} \cdots X_n^{i_n}$ coefficient of $\mathrm{ch}(V)$ is the rank of $\{v \in V : dv = X_1^i \cdots X_n^i \otimes v\}$.

If we compose ch with the restriction $\mathrm{R}_k(\mathrm{GL}_n) \to \mathrm{R}_k(D)$ we obtain a map that is called $\mathrm{ch}_G : \mathrm{R}_k(\mathrm{GL}_n) \to H$.

Theorem 11. *The homomorphism* ch_G *is injective. Its image is the subgroup* H^W *of* $H := \mathbb{Z}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$ *formed by the elements that are invariant under* $W = S_n$*, where* W *acts on* H *by permutation of the* X_i*.*

3.1 Injectivity

We will factor the map ch_G via the character group $X := \{\chi_V : \mathrm{GL}_n(k) \to k : V \text{ is a } G\text{-representation}\}$. In the previous section we have proved that the map from $\mathrm{R}_k(\mathrm{GL}_n) \to X$ is injective. Now we will consider why the map $X \to H$ is injective, proving that the composition is injective.

Proposition 7 of [2, p. 48] tells us that with each element of $f \in H$ corresponds a comodule, say E. For each monomial $m \in H$ the module E has an m-component of rank equal to the coefficient of m in f, f_m. We have $E = \bigoplus E_m$.

Let $D \subset \mathrm{GL}_n$ be the (diagonalizable) subgroup of diagonal matrices and let $M \in D(k)$ be an arbitrary element. Write $M = \mathrm{diag}(d_1, \ldots, d_n)$, then M acts on the m-component by multiplication with $f_m \cdot m(d_1, \ldots, d_n)$. In particular $\chi_E(M) = f(d_1, \ldots, d_n)$. The fact that $X \to H$ is injective follows from the fact that there is only one polynomial when we fix a set of values in all points of $(\mathbb{Z} \setminus 0)^n$.

3.2 Image

We will prove our result by proving the following two lemmas.

Lemma 12. *The image of* ch_G *is contained in* H^W*.*

Proof. As k is commutative, we have $\chi(AB) = \chi(BA)$ for all $A, B \in \mathrm{GL}_n(k)$. Let $\sigma \in S_n$ and consider the matrix P that permutes the standard basis by σ. Then for all $M \in D(k)$ as in the previous section, we have $\chi(PMP^{-1}) = \chi(M)$. In particular, in the terms of the proof in the last section, we must have $f(d_1, \ldots, d_n) = f(\sigma(d_1, \ldots, d_n))$. Hence, the polynomial $f \circ \sigma$ must be equal to the polynomial f for all $\sigma \in S_n$ and hence $f \in H^{S_n}$. □

Lemma 13. *The subset H^{S_n} is contained in the image of* ch_G.

Proof. Let $V = k^n$ and let $\mathrm{GL}_n(k)$ act on it by multiplication. It naturally extends to a $\mathrm{GL}_{n,k}$-module. Clearly $M = (d_1, \ldots, d_n) \in D(k)$ acts on the basis vectors e_i of V by multiplication with d_i. Hence, $\chi_V(d_1, \ldots, d_n) = d_1 + \ldots + d_n$ and $X_1 + \ldots + X_n$ is in the image of ch_G.

For the other symmetric polynomials s_i of degree i we consider $\bigwedge^i V$. The basis vectors are of the form $e_{j_1} \wedge \ldots \wedge e_{j_i}$ and M acts on it by multiplication with $d_{j_1} \cdot \cdots \cdot d_{j_i}$. This proves that $\chi_V(d_1, \ldots, d_n) = s_i(d_1, \ldots d_n)$ and hence s_i is in the image of ch_G.

As a ring H^{S_n} is generated by the s_i. As the character of a tensor product of representations is the product of characters, the lemma can now be considered to be proven. □

References

[1] J.S. Milne. *Basic Theory of Affine Group Schemes.* `http://www.jmilne.org/math/CourseNotes/`, 2013.

[2] Jean-Pierre Serre. Groupe de Grothendieck des schémas en groupes réductifs déployés. *Publications mathématiques l'I.H.É.S.*, vol. 34 (1968), p. 37 – 62.