Reduction of Plane Quartics and Cayley Octads

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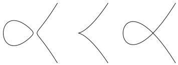
Reduction of elliptic curves and BSD

Consider an elliptic curve E over $\mathbb{Q}$. The reduction of E modulo p can be:

- elliptic curve E_p over $\mathbb{F}_p$: *good reduction*,
- projective line with a cusp: *additive reduction*,
- projective line with a node: *multiplicative reduction*.



Definition

The ***L*-function** of E is $L(E,s) = \prod_p L_p(p^{-s})^{-1}$, where

$$L_p(T) = \begin{cases} 1 - a_p T + pT^2 & \text{if } E \text{ has good reduction at } p \\ 1 \pm T & \text{if } E \text{ has multiplicative reduction at } p \\ 1 & \text{if } E \text{ has additive reduction at } p \end{cases},$$

and $a_p = p + 1 - |E_p(\mathbb{F}_p)|$.

Conjecture (Birch and Swinnerton-Dyer)

The L-function extends to a holomorphic function $\mathbb{C} \to \mathbb{C}$ *and has a zero of order* $\mathrm{rk}(E)$ *at* $s = 1$.

Stable curves

One can define the L-function for higher genus curves using their stable reduction.

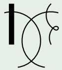
Definition

A **stable curve** is a connected reduced (but not irreducible) curve such that

- every singular point is an ordinary double point (node),
- every geometric compoment of genus 0 has at least 3 singular points, where self-intersections are counted twice,
- the arithmetic genus (sum of the genera of the components + the number of loops) of the curve is at least 2.

Example

In the following example, the thick line represents a component of genus 1 and the thin line one of genus 0. This is a stable curve of artihmetic genus 3.



Stable reduction

Theorem (stable reduction theorem)

Let C be a curve over a number field and let $\mathfrak{p}$ be a prime. After a finite extension of the base field, C admits a model over $\mathcal{O}_\mathfrak{p}$, such that the reduction modulo $\mathfrak{p}$ is a stable curve.

Example

Elliptic curves always obtain either good or multiplicative reduction after extending the base field. This depends on whether the j-invariant of the curve lies in $\mathcal{O}_\mathfrak{p}$ or not.

Note that the stable reduction does not change, if one extends the field even further.

Goal

Given a curve, determine its stable reduction.

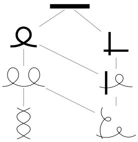
How many types of stable reduction?

Question

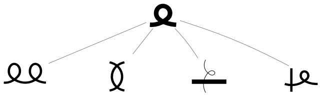
What are the combinatorial types of stable curves of a fixed arithmetic genus?

Starting from some stable type, there are two ways you can degenerate it:

- replace a component of genus g by a component of genus $g-1$ with a self-intersection;
- replace a component of genus g by two components of genera g_1 and g_2 such that $g_1 + g_2 = g$, and then redistribute the intersection points of the original component.

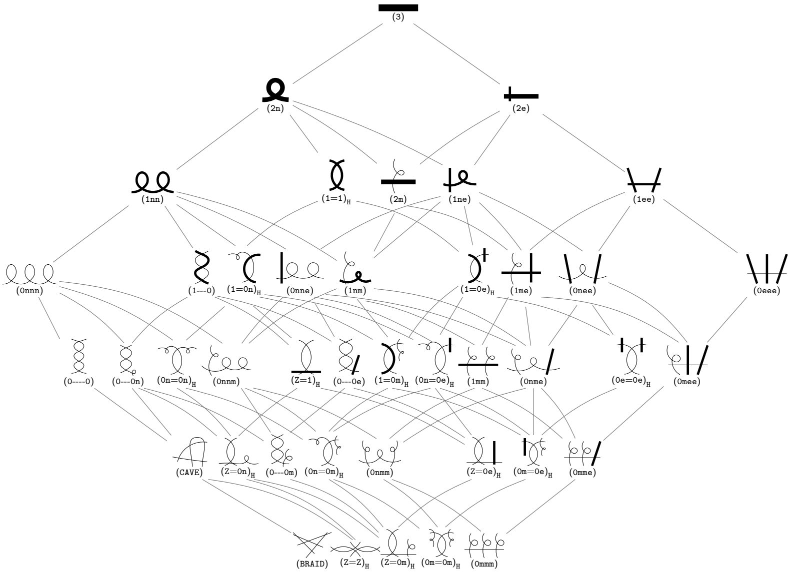


All genus 2 reduction types



Nodal arithmetic genus 3 curve degenerating

All 42 stable reduction types for genus 3 curves



Cluster pictures for hyperelliptic curves

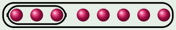
Dokchitser, Dokchitser, Maistret, and Morgan introduced the machinery of **cluster pictures** for hyperelliptic curves. The idea is to study the arithmetics of an hyperelliptic curve $y^2 = f(x)$ over $\mathbb{Q}_p$, for $p > 2$, by considering the p-adic distances between the roots of $f(x)$.

Example

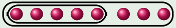
Let $p > 5$ and let

$$H\colon y^2 = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-p)(x-2p).$$

Three of the roots are p-adically closer to each other than to the rest of the roots, so H has the following **cluster picture**.



Replacing each root r by $\frac{1}{p+r}$, we obtain an equivalent cluster picture.



An example of stable reduction

While it was of course already known how to determine the stable reduction of a hyperelliptic curve, cluster pictures provide a convenient and conceptual way to think about it.

Example

Reducing the equation, modulo p, for

$$H\colon y^2 = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-p)(x-2p)$$

we get a **genus 2 curve with a cusp** at $(x, y) = (0, 0)$.

If we "zoom in" on the left cluster by substituting $x = px'$ and $y = p^{3/2}y'$ we get

$$y'^2 = x'(px'-1)(px'-2)(px'-3)(px'-4)(px'-5)(x'-1)(x'-2)$$

and the reduction is a **genus 1 curve with a cusp** at infinity.

It turns out the the **stable reduction** of H is the curve obtained by gluing these genus 1 and genus 2 curves at their cusps.

Cluster pictures and stable reduction

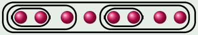
It is not hard to see what clusters of other sizes correspond to. The statement below for genus 3 can easiliy by generalised to any arbitrary genus.

Theorem

Let H be a genus 3 hyperelliptic curve and $\overline{H}$ the stable reduction of H modulo p. Then:

- *a* **cluster of size 4** *corresponds to a decomposition $\overline{H} = E_1 \cup E_2$ where E_1 and E_2 are curves of arithmetic genus 1 intersecting in two points;*
- *a* **cluster of size 3 or 5** *corresponds to a decomposition $\overline{H} = E \cup C$ where E and C are curves of arithmetic genus 1 and 2 intersecting in one point;*
- *a* **cluster of size 2 or 6** *corresponds to a node in $\overline{H}$ not described by a cluster of size 3, 4, or 5.*

Example

 corresponds to 

What is a Cayley octad?

We will define Cayley octads as a generalisation of the Weierstraß points of a hyperelliptic curve. This seems to work better than the 28 bitangents.

Consider a plane quartic $C\colon f(x,y,z) = 0$ in $\mathbb{P}^2$. It turns out there are essentially 36 ways[1] to write

$$f(x,y,z) = \det(xL + yM + zN),$$

where L, M, N are symmetric 4×4-matrices. Next, consider L, M, and N as quadratic forms q_L, q_M, and q_N in four variables.

Definition

The 8 intersection points of q_L, q_M, and q_N inside $\mathbb{P}^3$ form a **Cayley octad** O associated to C.

Proposition (see book Dolgachev-Ortland)

The points of O are **non-degenerate***: no two coincide, no three lie on a line, no four on a plane, and no seven on a twisted cubic.*
Moreover, the curve C is **uniquely determined** *by O.*

[1] The 36 ways correspond to the 36 even theta characteristics of C, i.e. the divisor classes D such that $2D \cong K_C$ and $\dim(\mathcal{L}(D))$ is even.

Reconstructing the plane quartic

- Start with 7 points A to G in $\mathbb{P}^3$ such that:
  - no two coincide;
  - no three lie on a plane;
  - no four lie on a plane;
  - no seven lie on a twisted cubic curve, i.e. the image of a degree 3 map $\mathbb{P}^1 \to \mathbb{P}^3$.
- In total, the space of quadrics in 4 variables has dimension 10, so we expect a 3-dimensional subspace $\langle q_1, q_2, q_3 \rangle$ to go through A to G.
- By Bézout's theorem, the three quadrics q_1, q_2, and q_3 intersect in 8 points. Let H be the eighth intersection point.
- The three quadrics parametrise a projective plane $\mathbb{P}^2$, and the singular quadrics in there form a curve: the plane quartic curve.

Degenerations of Cayley octads

Let $p > 3$. Just as the 8 Weierstraß points of a hyperelliptic curve could coincide modulo p, the Cayley octad can degenerate modulo p.

Remark

The 8 points in the Cayley octad are not independent; any point is uniquely determined by the other 7 points. As a consequence, the degenerations that can actually occur for O can be a bit more complicated. For example, if 4 of the points of O lie on a plane, this forces the other 4 points to also lie on a plane, in most cases[2].

These degenerations are related to the stable reduction of the curve.

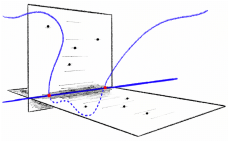
Theorem (see book Dolgachev-Ortland)

The points of O degenerate to 8 distinct points on a twisted cubic modulo p, if and only if C has good hyperelliptic reduction. Moreover, the 8 points on the twisted cubic are the 8 Weierstraß points of the hyperelliptic curve.

[2] There are more degenerate situations in which this is not true, e.g. when two points coincide.

Another degeneration of the Cayley octad

Consider the case in which four points (and the four complementary points) lie on a plane modulo p.



The Cayley octad, which also comes with an embedding[3] of C into $\mathbb{P}^3$, then degenerates to the picture above. We get a genus 2 curve and another component, which is the intersection of the two planes on which the 8 points lie. The components intersect twice.

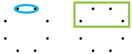
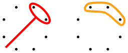
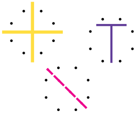
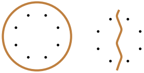
Corollary

In the situation described above, the stable reduction of C is a genus 2 curve with a node.

[3] This embedding comes from taking 'one and a half' times the canonical divisor.

Classifying degenerations

We introduce **four types of degenerations** which we conjecture to correspond to specific degenerations in the stable reduction of the curve.

type of block	corresponds to	pictures
α-blocks	clusters of size 2 or 6	
χ-blocks	clusters of size 3 or 5	
ϕ-blocks	clusters of size 4	
hyperelliptic blocks	hyperelliptic reduction	

Conjecture

The stable reduction of a plane quartic is determined by the above degenerations that occur in (any of) its Cayley octad(s).

Valuation data

We define valuation data to quantify the degenerations of a Cayley octad.

Definition

Let $\mathtt{A}$ to $\mathtt{H}$ be a Cayley octad in $\mathbb{P}^3$. Then we define its **valuation data** v as:

- for each of the $\binom{8}{4} = 70$ quadruples of points

  $$\mathtt{X} = (x_0 : x_1 : x_2 : x_3), \quad \mathtt{Y} = (y_0 : y_1 : y_2 : y_3)$$

  $$\mathtt{Z} = (z_0 : z_1 : z_2 : z_3), \quad \mathtt{W} = (w_0 : w_1 : w_2 : w_3)$$

  normalised such that $\min(\nu_K(x_0), \ldots, \nu_K(x_3)) = 0$, where ν_K is the valuation, and similarly for $\mathtt{Y}, \mathtt{Z}$, and $\mathtt{W}$, the integer

  $$v_{\mathtt{XYZW}} := \nu_K\left(\det\begin{pmatrix} x_0 & y_0 & z_0 & w_0 \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{pmatrix}\right),$$

  which measures the multiplicity with which $\mathtt{X}, \mathtt{Y}, \mathtt{Z}$, and $\mathtt{W}$ lie on a plane.
- an integer $v_\dagger$ measuring the multiplicity with which $\mathtt{A}$ to $\mathtt{H}$ lie on a twisted cubic.

Example valuation data

Example

Take $\mathtt{A} = (1:0:0:0)$, $\mathtt{B} = (1:p:0:0)$, and the other points generic. Then $v_{\mathtt{AB}\star\star} = 1$ and all other valuations are 0. We denote this valuation data by $v_{\mathrm{pt}}^{\mathtt{AB}}$.

Think of $v_{\mathrm{pt}}^{\mathtt{AB}}$ as a vector with 71 entries: one for each quadruple of octad points, and one entry for the twisted cubic index.

Remark

The geometry of the picture determines the valuation data, but the converse is very subtle. For example, it is not true that A and B collide if and only if all of the entries of $v - v_{\mathrm{pt}}^{\mathtt{AB}}$ are non-negative. *For example, you can put the 8 points on a plane without* A *and* B *colliding.*

Standard valuation vectors

Definition

We define:

- $v_{\mathrm{pt}}^{\cdots}$: valuation data corresponding to points coinciding;
- $v_{\mathrm{ln}}^{\cdots}$: valuation data corresponding to points lying on a line;
- $v_{\mathrm{pln}}^{\cdots}$: valuation data corresponding to points lying on a plane;
- v_{tc}: valuation data corresponding to $\mathtt{A}$ to $\mathtt{H}$ lying on a twisted cubic.

Remark

Just like in the case of cluster pictures, the valuation data will change if you change the coordinates, i.e. if you apply a PGL_3-transformation.

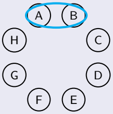
In our example, if we multiply the first coordinate by p, then the points $\mathtt{A}$ and $\mathtt{B}$ mapping to $(1:0:0:0)$ and $(1:1:0:0)$ no longer coincide. However, the other points will all end up on the plane $x=0$, e.g.

$$(1:2:3:4) \longmapsto (p:2:3:4),$$

so the new valuation data is $v_{\mathrm{pln}}^{\mathtt{CDEFGH}}$.

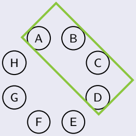
Alpha blocks (nodes)

Definition



We define:

- $\alpha_{1a}^{AB} = v_{pt}^{AB}$,
- $\alpha_{1b}^{AB} = v_{pln}^{CDEFGH}$.



We define:

- $\alpha_{2a}^{ABCD} = v_{pln}^{ABCD} + v_{pln}^{EFGH}$,
- $\alpha_{2b}^{ABCD} = v_{pln}^{ABCD} + v_{pt}^{ABCD}$.

Remark

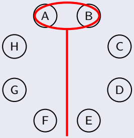
Note that α_{1a}^{AB} and α_{1b}^{AB} are PGL_3-equivalent, and similarly α_{2a}^{ABCD} and α_{2b}^{ABCD}.

The valuations $\alpha_{1\star}^{AB}$ and $\alpha_{2\star}^{ABCD}$ are not PGL_3-equivalent, but are related by a so-called Cremona transformation.

Out of the 36 octads, 16 will have a $\alpha_{1\star}$-block, and 20 will have a $\alpha_{2\star}$-block.

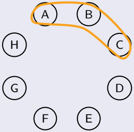
Chi blocks (genus 1 tails)

Definition



We define:

- $\chi_{1a}^{AB|CDE|FGH} = \max(v_{\ln}^{CDE} + v_{\ln}^{FGH}, v_{\mathrm{pln}}^{CDEFGH})$,
- $\chi_{1b}^{AB|CDE|FGH} = \max(v_{\mathrm{pt}}^{AB}, v_{\mathrm{pln}}^{ABCDE} + v_{\mathrm{pln}}^{ABFGH})$,
- $\chi_{1c}^{AB|CDE|FGH} = \max(v_{\ln}^{ABCDE} + v_{\ln}^{CDE}, v_{\mathrm{pt}}^{CDE})$.



We define:

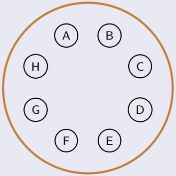
- $\chi_{2a}^{ABC} = v_{\ln}^{DEFGH} + v_{\mathrm{pln}}^{DEFGH}$,
- $\chi_{2b}^{ABC} = v_{\ln}^{ABC} + v_{\mathrm{pt}}^{ABC}$,
- $\chi_{2c}^{ABC} = v_{\ln}^{ABC} + v_{\mathrm{pln}}^{DEFGH}$.

Remark

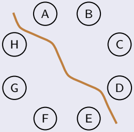
In this case, out of the 36 Cremona equivalent octads, 30 have a $\chi_{1\star}$-block and 6 have a $\chi_{2\star}$-block.

Hyperelliptic blocks

Definition



We define $\boldsymbol{TCu} = v_{\mathrm{tc}}$.



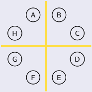
We define $\boldsymbol{Line} = v_{\mathrm{ln}}^{\mathtt{ABCD}}$.

Remark

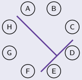
In this case, out of the 36 Cremona equivalent octads, 1 has a ***TCu***-block, and the other 35 have a ***Line***-block.

Phi blocks (1=1)

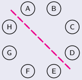
Definition



We define $\phi_{1a}^{\mathtt{AB|CD||EF|GH}} = v_{\mathrm{ln}}^{\mathtt{ABEF}} + v_{\mathrm{ln}}^{\mathtt{ABGH}} + v_{\mathrm{pln}}^{\mathtt{ABCD}} + v_{\mathrm{pln}}^{\mathtt{EFGH}}, \ldots,$



We define $\phi_{2b}^{\mathtt{ABC|FGH}} = v_{\mathrm{pt}}^{\mathtt{DE}} + v_{\mathrm{ln}}^{\mathtt{ABCD}} + v_{\mathrm{ln}}^{\mathtt{ABCE}}, \ldots,$



We define $\phi_{3b}^{\mathtt{ABCD}} = v_{\mathrm{pt}}^{\mathtt{ABCD}} + v_{\mathrm{ln}}^{\mathtt{ABCD}} + v_{\mathrm{pln}}^{\mathtt{ABCD}}, \ldots.$

Remark

In this case, out of the 36 Cremona equivalent octads, 18 have a $\phi_{1\star}$-block, 16 have a $\phi_{2\star}$-block, and 6 have a $\phi_{3\star}$-block.

One can think of a phi-block as a sum of an alpha-block and two hyperelliptic block.

Combining blocks

With cluster pictures of hyperelliptic curves one cluster is always contained in, containing, or completely disjoint from any other cluster. Similarly, there are rules for how Cayley octad pictures can be combined. For example, the pairs in the α-block can never overlap.

Theorem

The valuation data of any Cayley octad is an admissible sum of α-, ϕ-, χ-, and hyperelliptic blocks. Up to PGL_3-equivalence this sum is unique.

Proof sketch (work in progress).

Because the 8th point of the Cayley octad is determined by the first 7, the valuation data has certain restrictions. We study the space of possible valuation data using tools from tropical geometry, and through an extensive computation prove that they are all admissible sums of the building blocks we defined.

The uniqueness follows from an extensive linear algebra computation and can be found in our article and the accompanying code. □

Towards a proof (work in progress)

Remember how we "zoomed in" on a cluster in the case of hyperelliptic curves, to see a part of the stable reduction. Our proof strategy is similar, by zooming in on certain parts of the octad picture, it seems that we can see parts of the stable reduction.

Non-hyperelliptic case:

- one or two main components (depending on whether there is a χ-block), which has:
  - a cusp for every ϕ-block,
  - a node for every *visible* α-block,
  - a tacnode for the other main component, in case there is a χ-block,
- one genus 1 tail component for every ϕ-block, which has a very bad singularity at the place where the main component glues in, and it could have a node for a visible α-block.

Hyperelliptic case: the quartic could reduce to the square of a degree 2 function. We end up getting a toggle model, i.e.

$$Q^2 + p^s G = 0,$$

where Q mod p does not divide G mod p. The 8 intersection points of Q and G are the Weierstraß points of the hyperelliptic curve.

Overview

Algorithm (Main algorithm)

Input: plane quartic curve $C\colon f(x,y,z)=0$ and a prime p.
Output: stable reduction of C modulo p.

1. Compute a Cayley octad O for C.
2. Compute valuation data v for O.
3. Decompose v into building blocks.
4. Zoom in into different blocks to get different parts of the reduction.

Finding a Cayley octad

Algorithm (Plaumann-Sturmfels-Vinzant)

Input: plane quartic curve $C\colon f(x,y,z)=0$.
Output: a Cayley octad associated to C.

1. **Write down a generic line** $ax+by+cz=0$ **and find and solve the equations in** a,b,c **for the line to be a bitangent.** *This requires a big field extension!*
2. Pick three bitangents ℓ_1,ℓ_2,ℓ_3 going through points of bitangency $P_1,\ldots,P_6$ on C. We need $P_1+\ldots+P_6-K_C$ to be an even theta characteristic. If the next steps fail, try a different triple ℓ_1,ℓ_2,ℓ_3.
3. Compute the 4-dimension subspace V consisting of those elements of the 10-dimension space of cubics in x,y,z, that go through $P_1,\ldots,P_6$.
4. Let $v_{00}=\ell_1\ell_2\ell_3$, v_{01}, v_{02}, v_{03} be a basis of V. Find cubics v_{ij} such that $v_{0i}v_{0j}-v_{00}v_{ij}\equiv 0 \mod f$.
5. Let $V=(v_{ij})_{i,j=0}^{3}$. Now $M=f^{-2}\cdot V$ is a matrix of linear forms satisfying $\det(M)=\lambda\cdot f(x,y,z)$ for some constant λ. Use this to find a Cayley octad.

Finding the block decomposition

Algorithm

Input: valuation data v of a Cayley octad.
Output: decomposition of v as a sum of building blocks.

1. Find all building blocks B that occur in v, i.e. such that the entries of $v - B$ are non-negative.[a]
2. Construct all possible (maximal) subsets of these building blocks that are mutually compatible, i.e. such that the building blocks satisfy the rules that determine which ones can be combined.
3. For each subset, use linear algebra to check if v can be written as a linear combination of these building blocks.

[a] Note that there can be a lot of such building blocks. The fact that $v - B$ has non-negative entries does not guarantee that B occurs in the block decomposition of v.

Remark

Because of a large precomputation that we did, we know that v has a unique decomposition in building blocks. However, because of the large number of subsets that may occur in Step 2 of the algorithm, it can take some time to find this decomposition.

Challenges

Question

The algorithm requires to find a large field extension over which the bitangents can be defined. Is it possible to find the valuation data of the Cayley octad without computing in this field extension?

Question

Because of the large number of potential building blocks, the decomposition of v into building blocks is combinatorially expensive. Are there ways to make this faster?

Degenerations of theta characteristics

Definition

A **theta characteristic** on a curve C is a divisor class D such that $2D \cong K_C$, where K_C is the canonical divisor class.
Up to linear equivalence, there are $2^{2\text{genus}(C)}$ theta characteristics.

Question

How does this generalise to stable curves?

The answer to the question is: spin curve structures. I will not give a full definition, but will instead consider two examples.

Genus 2 glued to genus 1

Example

Consider a curve of the type  .

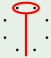
In this case, a spin curve structure consists of:

- a theta characteristic on the genus 2 curve (10 even and 6 odd),
- a theta characteristic on the genus 1 curve (3 even and 1 odd).

There are two ways to get an even theta characteristic:

- two even theta characteristics ($3 \cdot 10 = 30$ combinations),
- two odd theta characteristics ($1 \cdot 6 = 6$ combinations).

Remark

There are 30 Cayley octads with a  block, and 6 with a  block.

We conjecture that this corresponds exactly to these different types of spin curve structures.

Genus 2 with a node

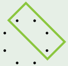
Example

Consider a curve of the type : a genus 2 curve with points P and Q glued.

Type A spin curve structure.
A divisor class D such that $2D \cong K_C + P + Q$. There are 32 such divisor classes and half of them are even, the other half is odd.

Type B spin curve structure.
Blow up the stable curve in the point $P = Q$. Take a line bundle corresponding to a theta characteristic on the genus 2 curve, and a degree 1 divisor on the $\mathbb{P}^1$ we just created by blowing up. Of these spin curve structures, 20 are even and 12 are odd (as for genus 2 curves).

Remark

There are 16 Cayley octads with a  block, and 20 with a  block.

We conjecture that this corresponds exactly to these different types of spin curve structures.

Correspondence between Cayley octads and spin curve structures

Conjecture

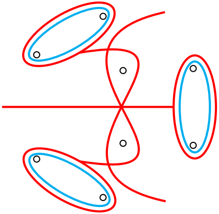
There is a one-to-one correspondence between the combinatorial types of Cayley octad pictures and the combinatorial types of spin curve structures on a stable curve.

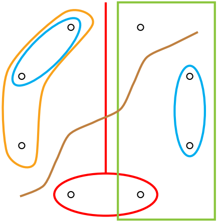
We have verified the conjecture combinatorially: the number of Cayley octad pictures of certain shapes and the number of combinatorial types of spin curve structures match up.

We currently have no proof that relates this to the curve.

Want to read more?

- Article: arXiV:2309.17381
- Magma package: github.com/rbommel/g3cayley





Hyperelliptic reduction types

In genus 2 all curves are hyperelliptic. In genus 3 (and higher) the situation is more complicated.

Definition

A stable curve C is called **hyperelliptic** if it admits an automorphism $\varphi\colon C \to C$ of order 2 such that:

- at each node fixed by φ, the action of φ has determinant 1, i.e. it either mirrors both components or keeps them both in the same orientation;
- the quotient C/φ is of arithmetic genus 0, i.e. a tree of $\mathbb{P}^1$s.

In genus 3 (and higher), there are reduction types that:

- are always hyperelliptic, e.g.  (take involutions swapping the two nodes on both genus 1 components),
- are never hyperelliptic, e.g.  (there is no space for three clusters of size 3 in a cluster picture with 8 roots),
- are sometimes hyperelliptic, in which case the locus of hyperelliptic curves has codimension 1 in the moduli space, e.g. in the case  (a genus 1 curve E with P glued to P' and Q glued to Q' is hyperelliptic if and only if $[P] + [P'] - [Q] - [Q'] = 0 \in \mathrm{Cl}(E)$).

Cremona transformations

There are 36 Cayley octads for each plane quartic curve. However, if you have one of these Cayley octads, you can find the others.

Definition

Let $\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}$ be four points of a Cayley octad O. Apply a PGL_3 transformation mapping these points to $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$, and $(0:0:0:1)$. The **Cremona transform** of O with respect to $\mathtt{ABCD}$ is the octad obtained by applying the transformation

$$(x:y:z:w) \mapsto \left(\tfrac{1}{x}:\tfrac{1}{y}:\tfrac{1}{z}:\tfrac{1}{w}\right)$$

to the other four points.

Note that the Cremona transforms with respect to $\mathtt{ABCD}$ and $\mathtt{EFGH}$ are PGL_3-equivalent.

Theorem

Suppose O is a Cayley octad associated to a plane quartic curve C. Then the 35 Cremona transforms of O are exactly the other 35 Cayley octads associated to C.