Explicit arithmetic intersection theory and computation of Néron-Tate heights

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Outline

This talk will consist of four parts

- introduction: the Néron-Tate height and its decomposition in local heights
- computation of the archimedean contribution of the Néron-Tate height
- computation of the non-archimedean contribution of the Néron-Tate height
- results: numerical verification of BSD in some new non-hyperelliptic cases

Generalised Birch and Swinnerton-Dyer conjecture

Tate has generalised the Birch and Swinnerton-Dyer conjecture to abelian varieties over number fields. We consider the case where J is the Jacobian of a curve C over $\mathbb{Q}$. Then the conjecture links:

- the special value of the L-function of J,
- the real period Ω,
- **the regulator** R,
- the Tamagawa numbers c_p,
- the size of $J(\mathbb{Q})_{\text{tors}}$,
- the (algebraic) rank r of $J(\mathbb{Q})$, and
- the size of the Tate-Shafarevich group $Ш(J)$,

through the formula: $\lim_{s\to 1}(s-1)^{-r}L(J,s) = \dfrac{\Omega \cdot R \cdot |Ш(J)| \cdot \prod_p c_p}{|J(\mathbb{Q})_{\text{tors}}|^2}$

Regulator

We know that $J(\mathbb{Q}) \cong \mathbb{Z}^r \times J(\mathbb{Q})_{\text{tors}}$ (Mordell-Weil).

Definition (regulator)

If $x_1, \ldots, x_r \in J(\mathbb{Q})$ are generators of the free part of $J(\mathbb{Q})$, then the *regulator* of $J(\mathbb{Q})$ is defined as

$$\left| \det \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \ldots & \langle x_1, x_r \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_r, x_1 \rangle & \langle x_r, x_2 \rangle & \ldots & \langle x_r, x_r \rangle \end{pmatrix} \right|,$$

where $\langle x_i, x_j \rangle = \frac{1}{2}(h_{\text{NT}}(x_i + x_j) - h_{\text{NT}}(x_i) - h_{\text{NT}}(x_j))$ is the height pairing associated to the Néron-Tate height on $J(\mathbb{Q})$.

Néron-Tate height

Identify each point of J with its inverse to obtain the Kummer variety $K = J/\pm$ associated to J. Let Θ be a Theta divisor on J. Then 2Θ descends to a very ample divisor on K, with an associated closed embedding $\iota\colon K \hookrightarrow \mathbb{P}^{2^g-1}$, where g is the genus of C.

Definition (Néron-Tate height)

We define a naive height $h_{\text{naive}}(x) = \log(\max(|x_1|, \ldots, |x_{2^g}|))$, where $(x_1 : \ldots : x_{2^g})$ are primitive integer coordinates for $\iota(x)$. The *Néron-Tate height* is then defined by:

$$h_{\text{NT}}(x) = \lim_{n\to\infty} \frac{1}{n^2} h_{\text{naive}}(nx) \qquad \text{for } x \in J(\mathbb{Q}).$$

Remark. *It is not practical to compute the Néron-Tate height using this definition.*

Local height contributions

Theorem (Faltings (1984), Hriljac (1985))

Let D and E be divisors on C of degree 0, with disjoint support. Then

$$h_{\mathrm{NT}}([D],[E]) = -\sum_{v} \langle D, E\rangle_v,$$

where we sum over all places, finite and infinite, of $\mathbb{Q}$.

The local heights $\langle D, E\rangle_v$ will be defined in the next two sections.

Note that $\langle D, E\rangle_v$ does depend on the specific choice of D and E, and does not define a pairing on $J(\mathbb{Q})$ (but their sum does).

Holmes (2012) and Müller (2014) already described algorithms to compute these local heights in the case C is hyperelliptic. Now we extend this to the general case.

Green's functions

Definition (Green's function)

Let E a divisor on C of degree 0, and let ω be a volume form. Then the *Green's function*

$$g_{E,\omega}\colon C(\mathbb{C}) \setminus \mathsf{supp}(E) \longrightarrow \mathbb{R}$$

is determined by the following properties:

- $g_{E,\omega}$ has a logarithmic singularity at $\mathrm{supp}(E)$,
- $dd^c g_{E,\omega} = \deg(E) \cdot \omega$, where $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{4\pi i}(\partial - \overline{\partial})$,
- $\int_C g_{E,\omega}\omega = 0$.

In order to compute the Green's function, we compute a period matrix for J, i.e. we realise $J_{\mathbb{C}}$ as $\mathbb{C}^g/\Lambda$, using code of Neurohr. The computation is then reduced to several evaluations of the classical Jacobi theta function. *Details omitted.*

Infinite local contribution

Definition (local pairing at infinite place)

Let $D = \sum_P n_P P$ be a divisor on C of degree 0, with support disjoint from E. Then

$$\langle D, E\rangle_\infty = \sum_P n_p g_{E,\omega}(P).$$

Remark. *The sum does not depend on ω, and defines a symmetric bilinear function on all pairs of divisors of degree 0 with disjoint support.*

Regular models

Definition (regular model)

Let p be prime. A *(regular) model* of C over $\mathbb{Z}_{(p)}$ is a (regular) integral, normal, projective flat $\mathbb{Z}_{(p)}$-scheme $\mathcal{C}$ of relative dimension 1, together with an isomorphism $\mathcal{C}_\eta \cong C$.

Example

The projective closure of the scheme $y^2 = x^3 + 3x^2 - 2x$ inside $\mathbb{P}^2$ over $\mathbb{Z}_{(2)}$ is a model for the curve over $\mathbb{Q}$ defined by the same equation. This model is not regular at the point $(0, 0)$ in the special fibre, i.e. at the maximal ideal $\mathfrak{m} = (x, y, 2)$, as all terms of the equation lie in $\mathfrak{m}^2$. In other words, the tangent space is too big.

By repeatedly blowing up, we can obtain a regular model.

Intersecting divisors on regular models

On a regular model $\mathcal{C}$, there are two types of divisors:

- horizontal divisors: closure of a divisor on the generic fibre $\mathcal{C}_{\mathbb{Q}}$;
- vertical divisors: divisors supported on the special fibre $\mathcal{C}_{\mathbb{F}_p}$.

These divisors can intersect.

Example

Let $\mathcal{C}$ be the projective closure of the scheme $y^2 = x^3 - 7x$ in $\mathbb{P}^2$ over $\mathbb{Z}_{(2)}$. Consider the closures $\mathcal{P}$ and $\mathcal{Q}$ of $(4,6) \in \mathcal{C}_{\mathbb{Q}}$ and $(4,-6) \in \mathcal{C}_{\mathbb{Q}}$. The horizontal divisors $\mathcal{P}$ and $\mathcal{Q}$ intersect in the point $(0,0) \in \mathcal{C}_{\mathbb{F}_2}$ with multiplicity

$$\text{length}\left(\frac{\left(\frac{\mathbb{Z}_{(2)}[x,y]}{y^2-x^3+7x}\right)_{(x,y,2)}}{(x-4,y-6)+(x-4,y+6)}\right) = \text{length}\left(\frac{\mathbb{Z}_{(2)}}{12}\right) = 2.$$

Intersection pairing on regular model

Definition (intersection number)

If $\mathcal{Q}$ and $\mathcal{R}$ are two distinct prime divisors on $\mathcal{C}$, then we define their *intersection number* as

$$\iota(\mathcal{Q}, \mathcal{R}) = \sum_{P \in \mathcal{Q} \cap \mathcal{R}} \text{multiplicity}_P(\mathcal{Q} \cap \mathcal{R}) \cdot \log |k(P)|,$$

where $k(P)$ is the residue field at P.

This extends to a bilinear function on all pairs of divisors on $\mathcal{C}$ with no common components.

Remark. *This does not respect linear equivalence. For example, the special fibre, which is a principal divisor, does have non-zero intersection with other divisors.*

Finite local contribution

Lemma

(a) *The function $\iota(\mathcal{D}, \mathcal{E})$ can be extended to all pairs of divisors, with $\mathcal{D}|_{\mathcal{C}_\eta}$ and $\mathcal{E}|_{\mathcal{C}_\eta}$ of degree 0 having disjoint support.*

(b) *Let D be a divisor of degree 0 on C. Then there exists a divisor $\Gamma(D)$ on the regular model $\mathcal{C}$, such that*

- *the horizontal part of $\Gamma(D)$ is the closure of D;*
- *$\iota(\Gamma(D), \mathcal{V}) = 0$ for each vertical divisor $\mathcal{V}$.*

Definition (local pairing at finite place)

Let D and E be two divisors on C of degree 0 with disjoint support. Then

$$\langle D, E\rangle_p = \iota(\Gamma(D), \Gamma(E)).$$

For the computation, we need to identify a finite set of p for which $\langle D, E\rangle_p$ is non-zero. *Details are omitted.*

Results

First result. We numerically verified the Birch and Swinnerton-Dyer conjecture, up to squares, for the split Cartan modular curve of level 13. This is a non-hyperelliptic curve of genus 3, whose Jacobian is of rank 3.

Second result. Let C be the projective closure of the scheme given by

$$3x^3y + 5xy^2 + 5y^4 - 5^9 = 0$$

inside $\mathbb{P}^2$, a curve with very bad reduction at 5. Consider the divisor $D = (1 : 0 : 0) - (0 : 25 : 1)$. We computed

$$h_{\mathrm{NT}}(D, D) \approx 3.2107.$$

Runtime. The first result took about 10 seconds of runtime. The second result took several minutes in `Magma`.