Kwack's theorem and its applications

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Abstract. In this talk we will consider Kwack's theorem, which generalises the big Picard theorem. We will look at some useful applications, including GAGA for maps from quasi-projective to projective hyperbolic varieties. Most of the content is based on [Kob98].

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1 Introduction

In complex analysis one of the most classical theorems is Liouville's theorem, which states that any bounded entire function $\mathbb{C} \to \mathbb{C}$ is constant. The result is improved by Picard's little theorem, which states that any such non-constant entire function can miss at most one value.

The big Picard theorem states that any holomorphic function with an essential singularity around $z = 0$, attains all but two values in $\mathbb{C}$ infinitely often in any neighbourhood of 0. It can also be phrased in terms of meromorphically extending certain functions:

Theorem 1 (Big Picard theorem)**.** *Let $f : \{0 < |z| < R\} \to \mathbb{C}$ be a holomorphic function from a punctured disc into $\mathbb{C}$. Suppose that f misses at most two values. Then f can be extended to a meromorphic function on $\{|z| < R\}$, i.e. a holomorphic function $\{|z| < R\} \to \mathbb{P}^1$.*

Remark 2. Missing just one point is not strong enough as $z \mapsto e^{1/z}$ has an essential singularity at 0, but misses the point 0.

The complex space $\mathbb{C} \setminus \{a, b\}$, for any distinct $a, b \in \mathbb{C}$, is (Kobayashi) hyperbolic. One might wonder if this statement is true in general.

Question 3. Let $Y \subset Z$ be a Kobayashi hyperbolic complex space inside a compact complex space. Does every function $\{0 < |z| < R\} \to Y$ extend to a function $\{|z| < R\} \to Z$?

The answer to this question is negative in general. For this to be true, one requires Y to be *hyperbolically imbedded* (and relatively compact) inside Z. The definition of this notion will not be treated in this talk, but can be found in [Kob98, Chap. 3, sect. 3] instead. Examples of hyperbolically embedded spaces are $\mathbb{C} \setminus \{a, b\} \subset \mathbb{P}^1$ and $X \subset X$ for any compact Kobayashi hyperbolic space X.

2 Kwack's theorem

Let $D^* = \{0 < |z| < 1\}$ be the punctured unit disc and let $D = \{|z| < 1\}$ be the unit disc.

The following theorem is due to Kwack [Kwa69]. The proof is based on the proof in [Kob98].

Theorem 4. *Let Y be a Kobayashi hyperbolic complex space, and let $f \colon D^* \to Y$ be holomorphic. Then f extends to a map $\overline{f} \colon D \to Y$ if there exists a sequence $z_k \in D^*$ converging to 0 such that $f(z_k)$ converges to a point $y_0 \in Y$.*[1]

Proof. First let $r_k = |z_k|$ and suppose w.l.o.g. that (r_k) is decreasing. We consider the circles parametrised by

$$\gamma_k(t) = f(r_k e^{2\pi i t}).$$

Let U be an open neighbourhood of y_0 inside Y, which we will identify, w.l.o.g., with

$$V = \left\{ \left|w^1\right| < \varepsilon, \cdots, |w^n| < \varepsilon \right\} \subset \mathbb{C}^n,$$

where y_0 is identified with 0. The goal is to show that some small open neighbourhood of "zero" in D^*, i.e. $\{|z| < \delta\} \subset D^*$ for certain $1 > \delta > 0$, is mapped into V by f. In this way, the function f can be extended continuously to a function $\overline{f}$ by setting $\overline{f}(0) = y_0$. This function will be automatically holomorphic due to Riemann's removable singularity theorem.

Let

$$W = \left\{ \left|w^1\right| < \tfrac{1}{2}\varepsilon, \cdots, |w^n| < \tfrac{1}{2}\varepsilon \right\}.$$

[1] This condition is automatically satisfied if Y is compact.

Holomorphic maps are distance decreasing. Therefore, the diameter of γ_k in the Kobayashi metric is at most

$$\frac{2\pi}{\log(1/r_k)},$$

which is going to 0 when $k \to \infty$. As Y was assumed to be hyperbolic, this implies that for k big enough, γ_k will be contained inside W. W.l.o.g., we will assume this to be the case for all k.

For each k, we can also consider the annulus $\{r_{k+1} < |z| < r_k\}$. If it is mapped into W for all but finitely many k, then we are done. So we will assume this not to be the case and w.l.o.g. we will assume all of these annuli to be not mapped inside W for all k.

For each k, we let

$$R_k = \{a_k < |z| < b_k\}$$

be the largest open annulus containing γ_k that maps into W. In particular, we get

$$r_{k+1} < a_k < r_k < b_k < r_{k-1}.$$

We let

$$\sigma_k(t) = a_k e^{2\pi i t}, \qquad \tau_k(t) = b_k e^{2\pi i t}$$

be the inner and outer boudary of R_k.

Claim 5. *There is a change of coordinates such that, for a certain large enough k, we have $f^1(z_k) \notin f^1(\sigma_k) \cup f^1(\tau_k)$, where f^1 is the first coordinate of f.*

Letting k be as in Claim 5, on the one hand, Cauchy's integral theorem yields

$$\int_{\sigma_k} \frac{df^1}{f^1(z) - f^1(z_k)} = \int_{f^1(\sigma_k)} \frac{dw^1}{w^1 - f^1(z_k)} = 0,$$

and, analogously,

$$\int_{\tau_k} \frac{df^1}{f^1(z) - f^1(z_k)} = \int_{f^1(\tau_k)} \frac{dw^1}{w^1 - f^1(z_k)} = 0.$$

On the other hand, as $\tau_k - \sigma_k$ (as signed sum of oriented curves) is the boundary of R_k, the residue theorem yields

$$\int_{\tau_k} \frac{df^1}{f^1(z) - f^1(z_k)} - \int_{\sigma_k} \frac{df^1}{f^1(z) - f^1(z_k)} = 2\pi i(Z - P),$$

where Z and P are the number of zeros and poles of $f^1(z) - f^1(z_k)$ in R_k, respectively. As $P = 0$ and $Z \geq 1$, this gives a contraction. $\square$

Proof of Claim 5. By definition of the annulus R_k both $f(\sigma_k)$ and $f(\tau_k)$ are contained in $\overline{W}$ but not in W. For each k, we choose points $p_k \in f(\sigma_k)$ and $q_k \in f(\tau_k)$ such that $p_k, q_k \in \partial W := \overline{W} \setminus W$. Then, w.l.o.g., taking a subsequence if necessary, we have $p_k \to p$ and $q_k \to q$ for certain points $p, q \in \partial W$. Making a linear change of coordinates, if necessary, we will assume that

$$w^1(p) \neq w^1(y_0) = 0 \neq w^1(q).$$

Moreover, as the diameter of $f(\sigma_k)$ and $f(\tau_k)$ in the Kobayashi metric go to 0, and Y is hyperbolic, we get that

$$\lim_{k\to\infty} f^1(\sigma_k) = w^1(p), \qquad \lim_{k\to\infty} f^1(\tau_k) = w^1(q), \qquad \lim_{k\to\infty} f^1(z_k) = w^1(y_0) = 0.$$

Therefore, for k sufficiently large, $f^1(z_k) \notin f^1(\sigma_k) \cup f^1(\tau_k)$. $\square$

3 Applications

In this section we will discuss a number of applications of Kwack's theorem. For these applications we first need to define what a meromorphic map of complex spaces is.

Definition 6. Let X and Y be complex spaces, and let $U \subset X$ be an open dense subset. Then a holomorphic map $f\colon U \to Y$ is called a *meromorphic map from X to Y* if the closure Γ of the graph of f inside $X \times Y$ is an analytic subset and the projection $\Gamma \to X$ is proper.

Remark 7. This notion extends to usual notion of meromorphicness for $Y = \mathbb{P}^1$.

Example 8. The function

$$f\colon \mathbb{C}^* \longrightarrow \mathbb{C}\colon z \longmapsto \tfrac{1}{z}$$

is not a meromorphic map from $\mathbb{C}$ to $\mathbb{C}$. If $\pi\colon \Gamma \to \mathbb{C}$ is the projection of the graph on the domain, then $\pi^{-1}(\{|z| \leq 1\}) \cong \{|z| \geq 1\}$ is not compact. However, f is meromorphic considered as map from $\mathbb{C}$ to $\mathbb{P}^1$.

Example 9. The function

$$f\colon \mathbb{C}^* \longrightarrow \mathbb{P}^1\colon z \longmapsto e^{1/z}$$

is not a meromorphic map from $\mathbb{C}$ to $\mathbb{P}^1$. The projection $\Gamma \to \mathbb{C}$ is proper. However, Γ is not an analytic subset of $\mathbb{C} \times \mathbb{P}^1$.

Theorem 10 ([Kob98, Thm. 6.3.19, p. 288])**.** *Let $f\colon X \to Y$ be a meromorphic map from a smooth complex space X into a Kobayashi hyperbolic complex space Y. Then f is holomorphic.*

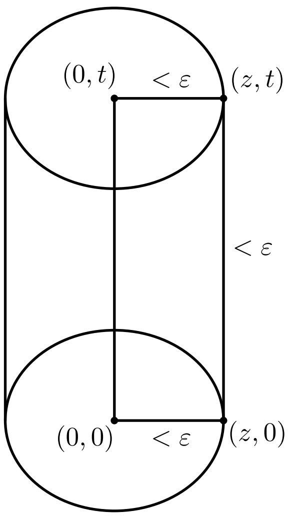
Proof. Let S be the complement of the maximum locus where f can be extended to a holomorphic map. Then S has at least codimension 1 everywhere, but S might not be smooth. Suppose that the theorem is proved for smooth S. Then we can extend our holomorphic map $X \setminus S \to Y$ to a holomorphic map $X \setminus S_{\text{sing}} \to Y$, which contradicts the definition of S. Therefore, it suffices to consider the case in which S is smooth.

In this case, we will assume w.l.o.g. that $X = D^m$ and $S \subset \{0\} \times D^{m-1}$. For each $t \in D^{m-1}$ we can consider the map

$$f_t \colon D^* \longrightarrow Y \colon z \longmapsto f(z,t).$$

The projection $\Gamma_f \to X$ is proper. Therefore, the graph Γ_f is compact. If z_k is a sequence in D^* converging to 0, then some subsequence of $((z_k, t), f_t(z_k))_k$ converges to a point $((0,t), y_t) \in \Gamma_f$. That means that $f_t(z_k) \to y_t$ for this subsequence and by Theorem 4 we can extend f_t to a holomorphic map $D \to Y$ by setting $f_t(0) = y_t$.

In order to prove that this new extended function $f \colon X \to Y$ is holomorphic, it suffices to show that it is continuous. By symmetry it suffices to do this at the point $(0,0) \in D \times D^{m-1}$. We can use the Kobayashi metric to prove this.



Side remark. *This proof is really uses hyperbolicity. If one takes the meromorphic function $f \colon D^* \times D \to \mathbb{P}^1 \colon (z,t) \mapsto \frac{t}{z}$, then the function $f_t(z) = \frac{t}{z}$ is holomorphic to $\mathbb{P}^1$ for all t, but*

$$t \longmapsto f_t(0) = \begin{cases} 0 & \textit{if } t = 0 \\ \infty & \textit{else} \end{cases}$$

is not continuous in t.

Let V (resp. W) be an open ball of radius ε around $0 \in D$ (resp. $0 \in D^{m-1}$) in the Kobayashi metric. Let $(z,t) \in D^* \times (D^{m-1} \setminus \{0\})$. Since both f_0 and f_t are holomorphic, we have

$$d_Y\left(f(z,t), f(0,t)\right) \leqslant d_D(z,0) < \varepsilon \quad \text{and} \quad d_Y\left(f(z,0), f(0,0)\right) \leqslant d_D(z,0) < \varepsilon.$$

As $f|_{D^* \times D^{m-1}}$ is holomorphic, we have

$$d_Y(f(z,0), f(z,t)) < d_{D^* \times D^{m-1}}((z,0),(z,t)) = d_{D^{m-1}}(0,t) < \varepsilon.$$

In any case, $d_Y(f(0,0), f(v,w)) < 3\varepsilon$ for all $(v,w) \in V \times W$. This proves that f is continuous at $(0,0)$. $\square$

This allows us to extend meromorphic functions as well, even in the case X is not smooth.

Theorem 11 ([Kob98, Thm. 6.3.24, p. 290])**.** *Let Y be a compact Kobayashi hyperbolic space. Let A be a closed complex subspace pure of codimension 1 inside a complex space X. Then every meromorphic map $f\colon X \setminus A \to Y$ extends to a meromorphic map $\overline{f}\colon X \to Y$.*

Proof. In the proof of Theorem 10 we only used that the graph of the function is proper over the domain. In our case the closure of the graph of f inside $X \times Y$ is proper over X as Y is compact. Therefore, Theorem 10 applies and f is holomorphic on X. □

Remark 12. The result is not obvious. For example, the meromorphic function $f\colon \mathbb{C} \to \mathbb{P}^1\colon z \mapsto e^z$ does not expand to a meromorphic function $\mathbb{P}^1 \to \mathbb{P}^1$. The hyperbolicity is really needed.

This result can be generalised to the case in which $Y \subset Z$ is hyperbolically imbedded.

Theorem 13. *Let $Y \subset Z$ be hyperbolically imbedded. Let $A \subset X$ be a closed subspace of codimension 1. Then every meromorphic map $X \setminus A \to Y$ extends to a meromorphic map $X \to Z$.*

Proof sketch. The idea is to first perform a resolution of singularities to get a smooth $\widetilde{X}$ with a normal crossings divisor $\widetilde{A} \subset \widetilde{X}$, biholomorphic to $A \subset X$. Then we can assume $\widetilde{X} = D^n$ and $\widetilde{X} \setminus \widetilde{A} = (D^*)^k \times D^{n-k}$. As $\overline{Y} \subset Z$ is compact, Theorem 10 can be used to extend to a map $\widetilde{X} \to Z$. In order to prove that this map is holomorphic, i.e. to generalise the proof of Theorem 11, one needs to use properties of hyperbolically imbedded spaces. This holomorphic map then gives a meromorphic map $X \to Z$. □

Example 14. By taking $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $Z = \mathbb{P}^1$, $X = D$ and $A = \{0\} \subset D$, we obtain the big Picard theorem.

In particular, the theorem also implies that the result of GAGA also holds for morphisms of quasi-projective varieties to projective Kobayashi hyperbolic varieties.

Corollary 15. *Let Y be a projective Kobayashi hyperbolic variety and let X be a quasi-projective variety. Then $Hom(X, Y) = Hom(X^{an}, Y^{an})$.*

Proof. Let $\overline{X}$ be the projective closure of X. Then

$$\operatorname{Hom}(X, Y) \longleftarrow \operatorname{Hom}\big(\overline{X}, Y\big) \overset{\text{GAGA}}{=} \operatorname{Hom}\big(\overline{X}^{\text{an}}, Y^{\text{an}}\big) \overset{\text{Theorem 11}}{=} \operatorname{Hom}(X^{\text{an}}, Y^{\text{an}}).$$

Therefore, the map $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(X^{\text{an}}, Y^{\text{an}})$ is bijective. □

References

[Kob98] S. Kobayashi, *Hyperbolic complex spaces. Spinger-Verlag*, 1998.

[Kwa69] M. H. Kwack, Generalization of the big Picard theorem. *Ann. of Math. (2)* **90**, 1969, 9–22.