Boundedness in families of projective varieties with applications to arithmetic hyperbolicity

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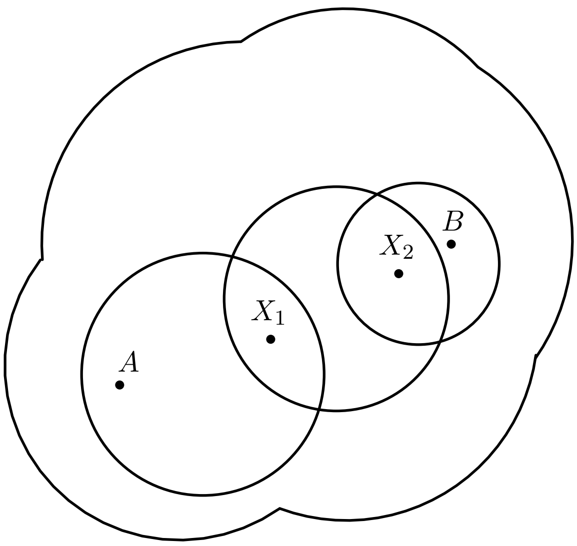
Abstract. There are several different notions of hyperbolicity, which are conjectured to be equivalent by Demailly, Green–Griffiths, Lang, and Vojta. We study how these notions behave in families of varieties, which provides evidence for the conjectures. Moreover, we study a new weaker notion of hyperbolicity, called mildly boundedness, and use it to prove new finiteness results for points on certain surfaces and semi-abelian varieties.

1 Introduction

1.1 Complex geometric notions



On each complex manifold X there is a pseudometric, called the Kobayashi pseudometric, obtained as follows. For two points $A, B \in X(\mathbb{C})$ we look at all paths of intersecting circles from A to B, and add the hyperbolic distances (e.g. from A to X_1 in the first circle, from X_1 to X_2 in the second circle, et cetera). The Kobayashi distance is the infimum of the sum of the hyperbolic distances for all such paths. In case this defines a metric, X is called *Kobayashi hyperbolic.*

On the other hand, a complex manifold X is called *Brody hyperbolic* if there are no non-constant holomorphic maps from $\mathbb{C}$ to X. For example, $\mathbb{C}$, $\mathbb{C}^*$, abelian varieties, blow-ups of smooth varieties, and $\mathbb{P}^2 \setminus \bigcup_{i=1}^4 \ell_i$, where $\ell_1, \ldots, \ell_5$ are lines in general position, are <u>not</u> Brody hyperbolic. Curves of genus at least 2, $\mathbb{C}^* \setminus \{1\}$, and $\mathbb{P}^2 \setminus \bigcup_{i=1}^5 \ell_i$ are Brody hyperbolic.

When X is compact, the notions are equivalent.

Theorem 1 (Brody)**.** *A compact complex manifold is Kobayashi hyperbolic iff it is Brody hyperbolic.*

1.2 Algebraic notions

Unless otherwise stated, k is a field of characteristic 0, $\overline{k}$ is its algebraic closure, and curves/varieties are defined over $\overline{k}$.

There are also some algebraic notions of hyperbolicity. Here we need X to be a projective variety over $\overline{k}$. We fix an ample line bundle $\mathcal{L}$ on X and consider morphisms φ from algebraic curves C to X. We say X is

(a) *algebraically hyperbolic*, if $\deg(\varphi^* \mathcal{L})$ is bounded linearly in genus(C);

(b) *1-bounded*, if $\deg(\varphi^* \mathcal{L})$ is bounded by a constant only depending on C;

(c) *bounded*, if for any irreducible variety V, there are only finitely many polynomials occurring as the Hilbert polynomial of $\tau^* \mathcal{L}$ with $\tau \colon V \to X$;

(d) *groupless*, if X does not admit a non-constant morphism from a positive dimensional algebraic group;

(e) *of very general type*, if every closed subvariety of X is of general type;

(f) *arithmetically hyperbolic*, if for every finitely generated subring $A \subset \overline{k}$ and every model $\mathcal{X}/A$ with $\mathcal{X}_{\overline{k}} \cong X$, the set of points $\mathcal{X}(A)$ is finite.

For example, a projective curve satisfies either of these conditions iff its genus is at least 2. Also, if X and Y both have one (or more) of the hyperbolicity properties, then their product $X \times Y$ has the same property.

Conjecture (Demailly, Green–Griffiths, Lang, Vojta)**.** *The six notions (a) to (f) are equivalent. Moreover, over the field $\mathbb{C}$, the notion is conjectured to be equivalent to X being Kobayashi/Brody hyperbolic.*

Partial results are known. For example, it is known that (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) and (e), (f) $\Rightarrow$ (d). In some special cases the conjecture is known.

Theorem (Bloch-Kawamata-Ochiai, Faltings, Kawamata-Ueno)**.** *Let X be a closed subvariety of an abelian variety over $\overline{k}$. Then the full conjecture holds for X.*

1.3 Properties of hyperbolicity

A subset of a noetherian scheme S is called *Zariski-countable closed* if it is a countable union of Zariski closed subsets. This defines the Zariski-countable topology on S. Our first result is the following.

Theorem 2 (Demailly, Nakayama, Javanpeykar-Vezzani, Van Bommel-Javanpeykar-Kamenova)**.** *Let S be a noetherian scheme over $\mathbb{Q}$, and let X/S be projective. Then the set of $s \in S$ such that the geometric fibre $X_{\bar{s}}$ is algebraically hyperbolic/(1-)bounded/groupless/of very general type is Zariski-countable open.*

It is unknown whether (some of) these loci are actually Zariski open. It is known that the Kobayashi hyperbolic locus is open in the complex analytic topology for compact families.

We know the following: if X has one of the first five properties over $\overline{k}$ and $k \subset L$, then $X_{\overline{L}}$ has the same property. For arithmetic hyperbolicity this is not known.

Question: if X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$, is it then also arithmetically hyperbolic over $\mathbb{C}$? Or, equivalently, if X has "finitely many points over number rings", does X have "finitely many points over any finitely generated ring"?

Our second result gives a positive answer for certain surfaces.

Theorem 3. *Let X be an arithmetically hyperbolic surface over $\overline{\mathbb{Q}}$ admitting a morphism to some abelian variety, such that the image is 2-dimensional. Then $X_{\mathbb{C}}$ is arithmetically hyperbolic.*

The proof makes uses of the notion of mildly bounded varieties, which will be explained in the next part.

2 Mildly bounded varieties

A variety X over k is called mildly bounded if, for every smooth quasi-projective connected curve C, there are points $c_1, \ldots, c_m \in C(\overline{k})$ such that, for every $x_1, \ldots, x_m \in X(\overline{k})$ the set

$$\mathrm{Hom}((C, c_1, \ldots, c_m), (X, x_1, \ldots, x_m))$$

of m-pointed morphisms is finite.

A non-example is $\mathbb{A}^1$, which is not mildly bounded as there exist morphisms of arbitrary high degree using Lagrange interpolation.

The following result gives rise to quite a few examples.

Theorem 4. *If X maps quasi-finitely to its Albanese variety, then X is mildly bounded.*

In particular, the following varieties are mildly bounded:

- all smooth connected curves except $\mathbb{P}^1$ and $\mathbb{A}^1$;

- all semi-abelian varieties;
- anything with a quasi-finite map to such a variety.

Proof sketch. Without loss of generality we can shrink C and assume C maps injective into its Albanese variety $\mathrm{Alb}(C)$. Then any morphism $C \to X$ gives rise to a morphism $\mathrm{Alb}(C) \to \mathrm{Alb}(X)$, and for any $c \in C(\overline{k})$ and $x \in X(\overline{X})$, the induced map

$$\mathrm{Hom}_{\mathrm{PtdVar}/\overline{k}}((C,c),(X,x)) \to \mathrm{Hom}_{\mathrm{AlgGrp}/\overline{k}}(\mathrm{Alb}(C),\mathrm{Alb}(X))$$

has finite fibres. Hence, it suffices to show that we can find $d_1,\ldots,d_\ell \in \mathrm{Alb}(C)(\overline{k})$ such that, for any $y_1,\ldots,y_\ell \in \mathrm{Alb}(X)(\overline{k})$, the set

$$\mathrm{Hom}_{\mathrm{AlgGrp}/\overline{k}}((\mathrm{Alb}(C),d_1,\ldots,d_\ell),(\mathrm{Alb}(X),y_1,\ldots,y_\ell))$$

is finite. We can then choose $d_1,\ldots,d_\ell$ such that they generate a dense subgroup of $\mathrm{Alb}(C)$. □

We also proved that the mildly bounded locus of a family of varieties is Zariski-countable open. As a consequence we get the following corollary.

Corollary 5. *Let X be a mildly bounded projective variety over k. If $k \subset L$ has finite transcendence degree, then X_L is mildly bounded.*

Proof sketch. The idea is to find a variety S over k with function field L. Then we spread out X_L to a projective scheme $\mathcal{X} \to S$. Its special fibres are mildly bounded by assumption, hence its generic fibre X_L is also mildly bounded by using the Zariski-countable openness of the locus. □

In fact, we conjecture the following.

Conjecture 6. *Let X be a projective variety over $\overline{k}$. Then X is mildly bounded if and only if there is no non-constant map from $\mathbb{P}^1$ to X.*

We can actually prove this for certain surfaces.

Theorem 7. *Let X be a projective integral surface not containing a rational line. Suppose X admits a morphism to an abelian variety A, and that the image of this morphism has dimension 2. Then X is mildly bounded over k.*

Proof sketch. In $\mathrm{Hom}(C,X)$, we distinguish between two types of morphisms: those which map to a point p in A, and those that do not. Those of the first type will be mildly bounded, because the fibre of p in X is a curve, which is then forced to be of genus at least 1 and hence mildly bounded. Those of the second type will be mildly bounded, because A is mildly bounded. □

Using the following lemma of Javanpeykar, we can prove Theorem 3.

Lemma (Javanpeykar)**.** *Suppose X is mildly bounded over k, and $k \subset L$ is an extension. Then X is arithmetically hyperbolic iff X_L is arithmetically hyperbolic.*

Proof of Theorem 3. As X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ by assumption, and mildly bounded by Theorem 7, the result follows from the lemma. □

Another result is that semi-abelian varieties over hyperbolic curves are mildly bounded.

Theorem 8. *Let S be a hyperbolic curve over k and let $\mathcal{X} \to S$ be a semi-abelian scheme of relative dimension d. Then $\mathcal{X}$, considered as $d+1$-dimensional scheme over k, is mildly bounded.*

The proof uses Silverman's specialization theorem, which states the following.

Theorem (Silverman's specialization theorem)**.** *Let A be an abelian variety of the function field K of a curve C. Let $\sigma_1, \dots, \sigma_n$ be independent sections of $A(K)$. Then there is a point in $p \in C(\overline{k})$, such that the specializations of $\sigma_1, \dots, \sigma_n$ to A_p are still independent.*

3 Pseudo-hyperbolicity

Take a curve C of genus at least 2, then C and also $C \times C$ is hyperbolic. If we then blow up $C \times C$ in one point, the resulting variety $X = \mathrm{Bl}_p(C \times C)$ is not hyperbolic as it contains a $\mathbb{P}^1$. Hence, hyperbolicity is not a birational invariant.

However, X is almost hyperbolic in the sense that the failure is concentrated in the exceptional locus Δ of the blow-up. We say that X is *algebraically/arithmetically hyperbolic/bounded modulo* Δ. In the definition we only consider morphisms $\varphi\colon C \to X$ that do not land completely in Δ, and points of $\mathcal{X}(A)$ that do not lie in Δ.

We then say X is *pseudo-algebraically/arithmetically hyperbolic* (resp. pseudo-bounded) if there is a closed $\Delta \subsetneq X$ such that X is algebraically/arithmetically hyperbolic (resp. bounded) modulo Δ.

Conjecture (Lang, Vojta)**.** *Let X be projective. Then X is pseudo-hyperbolic if and only if X is of general type.*

In particular, we expect this to be stable under base change, which is our last result.

Theorem 9. *Assume X is projective over k, X is pseudo-algebraically hyperbolic, and $k \subset L$ is an extension. Then X_L is pseudo-algebraically hyperbolic.*

4 One proof

First we sketch a proof of the following result.

Theorem. *Let S be noetherian over $\mathbb{Q}$ and X/S projective. Then the algebraically hyperbolic locus in S is Zariski-countable open.*

Proof sketch. Let $\mathcal{U}_g \to \mathcal{M}_g$ be the universal curve over the algebraic stack of smooth curves of genus g. Let

$$\mathcal{H}_{g,d} = \underline{\mathrm{Hom}}_{\mathcal{M}_g}(\mathcal{U}_g, X \times \mathcal{M}_g)$$

be the stack of morphisms of degree d with respect to $\mathcal{L}$ from curves to X.

Let $S_{g,d}$ be the stack-theoretic image of the structure map $\mathcal{H}_{g,d}$. These are exactly the points in s for which there is a map from a genus g curve defined over k_s, to the fibre X_s. For a fixed β we define

$$S_\beta = \bigcup_{d > \beta \cdot g} S_{g,d},$$

the locus where there is a map of degree greater than $\beta \cdot g$.

A priori, S_β is a countable union of locally closed subsets of S. We then use a classical result from algebraic geometry to prove that S_β is Zariski-countable close. In particular, the non-algebraically hyperbolic locus

$$S^{\mathrm{non-hyp}} = \bigcap_{\beta=1}^{\infty} S_\beta$$

is also Zariski-countable closed. $\square$