Reduction of Plane Quartics and Cayley Octads

Raymond van Bommel

Massachusetts Institute of Technology
Simons Collaboration on Arithmetic Geometry, Number Theory, and Computation

joint work with
Jordan Docking, Vladimir Dokchitser,
Elisa Lorenzo García, Reynald Lercier

11 January 2024

Massachusetts Institute of Technology

Cluster pictures for hyperelliptic curves

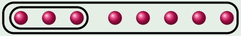
Dokchitser, Dokchitser, Maistret, and Morgan introduced the machinery of **cluster pictures** for hyperelliptic curves. The idea is to study the arithmetics of an hyperelliptic curve $y^2 = f(x)$ over $\mathbb{Q}_p$, for $p > 2$, by considering the p-adic distances between the roots of $f(x)$.

Example

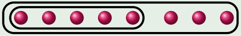
Let $p > 5$ and let

$$H\colon y^2 = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-p)(x-2p).$$

Three of the roots are p-adically closer to each other than to the rest of the roots, so H has the following cluster picture.



Replacing each root r by $\frac{1}{p+r}$, we obtain an equivalent cluster picture.



An example of stable reduction

After extending the base field if necessary, any curve of genus at least 2 attains **stable reduction**. We will determine this reduction for H.

Example

Reducing the equation, modulo p, for

$$H\colon y^2 = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-p)(x-2p)$$

we get a **genus 2 curve with a cusp** at $(x, y) = (0, 0)$.

If we "zoom in" on the left cluster by substituting $x = px'$ and $y = p^{3/2}y'$ we get

$$y'^2 = x'(px'-1)(px'-2)(px'-3)(px'-4)(px'-5)(x'-1)(x'-2)$$

and the reduction is a **genus 1 curve with a cusp** at infinity.

It turns out the the **stable reduction** of H is the curve obtained by gluing these genus 1 and genus 2 curves at their cusps.

Cluster pictures and stable reduction

It is not hard to see what clusters of other sizes correspond to.

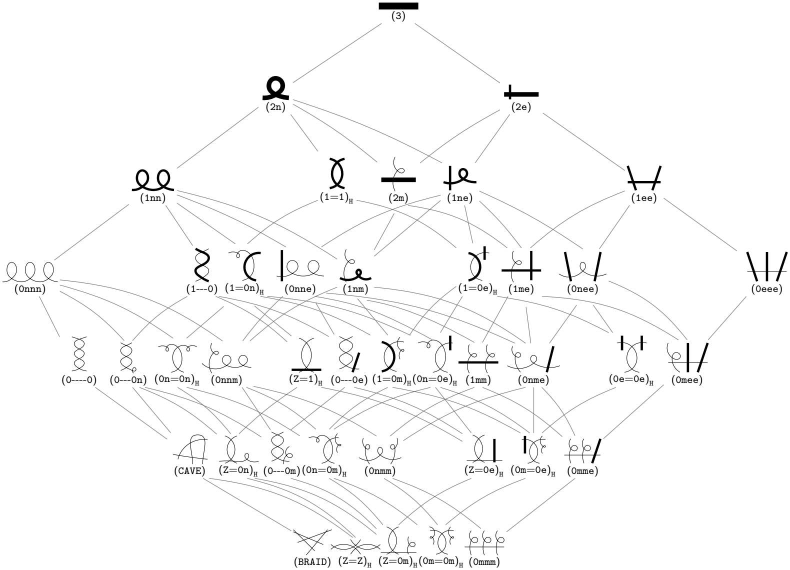
Theorem (Dokchitser-Dokchitser-Maistret-Morgan)

Let H be a genus 3 hyperelliptic curve and $\overline{H}$ the stable reduction of H modulo p. Then:

- *a* **cluster of size 4** *corresponds to a decomposition $\overline{H} = E_1 \cup E_2$ where E_1 and E_2 are curves of arithmetic genus 1 intersecting in two points;*
- *a* **cluster of size 3 or 5** *corresponds to a decomposition $\overline{H} = E \cup C$ where E and C are curves of arithmetic genus 1 and 2 intersecting in one point;*
- *a* **cluster of size 2 or 6** *corresponds to a node in $\overline{H}$ not described by a cluster of size 3, 4, or 5.*

Moreover, these clusters can be combined to create more complicated stable curves.

All 42 stable reduction types for genus 3 curves



What is a Cayley octad?

Now consider a plane quartic $C\colon f(x,y,z)=0$ in $\mathbb{P}^2$. It turns out there are essentially 36 ways[1] to write

$$f(x,y,z)=\det(xL+yM+zN),$$

where L, M, N are symmetric 4×4-matrices. Next, consider L, M, and N as quadrics q_L, q_M, and q_N in four variables.

Definition

The 8 intersection points of q_L, q_M, and q_N inside $\mathbb{P}^3$ form a **Cayley octad** O associated to C.

Proposition (see book Dolgachev-Ortland)

The points of O are **non-degenerate**: *no two coincide, no three lie on a line, no four on a plane, and no seven on a twisted cubic. Moreover, the curve C is* **uniquely determined** *by O.*

[1] The 36 ways correspond to the 36 even theta characteristics of C.

Degenerations of Cayley octads

Let $p > 3$. Just as the 8 Weierstraß points of a hyperelliptic curve could collide modulo p, the Cayley octad can degenerate modulo p.

Remark

The 8 points in the Cayley octad are not independent; any point is uniquely determined by the other 7 points. As a consequence, the degenerations that can actually occur for O can be a bit more complicated. For example, if 4 of the points of O lie on a plane, this forces the other 4 points to also lie on a plane, in most cases[2].

These degenerations are related to the stable reduction of the curve.

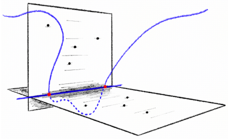
Theorem (see book Dolgachev-Ortland)

The points of O degenerate to 8 distinct points on a twisted cubic modulo p, if and only if C has good hyperelliptic reduction. Moreover, the 8 points on the twisted cubic are the 8 Weierstraß points of the hyperelliptic curve.

[2] There are more degenerate situations in which this is not true, e.g. when two points collide.

Another degeneration of the Cayley octad

Consider the case in which four points (and the four complementary points) lie on a plane modulo p.



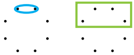
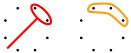
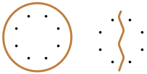
The Cayley octad, which also comes with an embedding of C into $\mathbb{P}^3$, then degenerates to the picture above. We get a genus 2 curve and another component, which is the intersection of the two planes on which the 8 points lie. The components intersect twice.

Corollary

In the situation described above, the stable reduction of C is a genus 2 curve with a node.

Classifying degenerations

We introduce **four types of degenerations** which we conjecture to correspond to specific degenerations in the stable reduction of the curve.

type of block	corresponds to	pictures
α-blocks	clusters of size 2 or 6	
χ-blocks	clusters of size 3 or 5	
ϕ-blocks	clusters of size 4	
hyperelliptic blocks	hyperelliptic reduction	

Conjecture

The stable reduction of a plane quartic is determined by the above degenerations that occur in (any of) its Cayley octad(s).

Evidence

Just as clusters must be contained in or disjoint from each other, we provide rules which dictate how blocks may be combined into an **octad picture**. When you change the coordinates of $\mathbb{P}^3$, the degenerations that you get may change, but the octad picture cannot.

Theorem

The construction of the octad picture does not depend on the choice of coordinates in $\mathbb{P}^3$.

Moreover, we have an **explicit procedure**

$$\{\text{octad pictures}\} \to \{\text{stable reduction types}\}.$$

Theorem

The above procedures gives an almost-bijection between the equivalence classes of octad pictures and the stable reduction types.

We computationally verified this against a dataset of thousands of plane quartics, in which each of the 42 stable reduction types occurs.

Want to read more?

- Preprint: arXiV:2309.17381
- Magma package: github.com/rbommel/g3cayley