# Néron models

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## Motivation and definition

Given an elliptic curve $E$ defined over $\mathbb{Q}$ and a prime $p \in \mathbb{Z}$, there is a good notion of reduction of $E$ modulo $p$. Namely, among all possible Weierstrass equations for $E$ with integer coefficients, one chooses the model $\mathcal{E}$ which minimizes the $p$-adic valuation of the discriminant (this is called the minimal Weierstrass model for $E$ at $p$) and then reduces the equation modulo $p$, producing a curve $\overline{E}$ of genus one over $\mathbb{F}_p$. We also have a notion of reduction modulo $p$ for points of the elliptic curve, i.e. there is a well-defined group homomorphism

$$E(\mathbb{Q}) \to \overline{E}(\mathbb{F}_p)$$

obtained by lifting a $\mathbb{Q}$-valued point of $E$ to a $\mathbb{Z}_{(p)}$-valued point of $\mathcal{E}$ and then reducing modulo $p$.

It is well known that for any elliptic curve over $\mathbb{Q}$ there are primes $p$ such that the reduction modulo $p$ is a singular curve. These are called primes of bad reduction. A drawback of bad reduction is that the group structure of $E/\mathbb{Q}$ does not carry over to the reduction $\overline{E}/\mathbb{F}_p$. Indeed, if a variety over a field has a group structure, it is necessarily smooth: intuitively, the automorphisms defined by translation by points make the variety look everywhere the same. One way around this is to remove the singular points of $\overline{E}$. The resulting variety $\overline{E}^{sm}$ has a group structure, which we obtain at the expense of projectivity.

Let's look at an example: consider the elliptic curve given by $y^2 = x^3+3x-4$ over $\mathbb{Q}$. There are different types of reductions at different primes. At the prime 7 the reduction is good; the curve defined by this equation over $\mathbb{F}_7$ is an elliptic curve. At the prime 5 the reduction is multiplicative (a node); the curve defined by this equation over $\mathbb{F}_5$ has a singular point and if we remove it the remaining points have the structure of the group $\mathbb{F}_5^*$. At the prime 3 the reduction is additive (a cusp); the curve defined by this equation over $\mathbb{F}_3$ has a singular point and if we remove it the remaining points have the structure of the additive group of $\mathbb{F}_3$.

The variety $\overline{E}^{sm}$ obtained by removing the non-regular points still has one serious drawback: we have lost along the way the reduction of points mod $p$. Indeed, in general we have no map $E(\mathbb{Q}) \to \overline{E}^{sm}(\mathbb{F}_p)$, as the reduction map $E(\mathbb{Q}) \to \overline{E}(\mathbb{F}_p)$ need not factor via $\overline{E}^{sm}(\mathbb{F}_p)$.

Here is an example: Consider the elliptic curve $E/\mathbb{Q}$ given by Weierstrass equation $y^2 - x^3 + x^2 + 49 = 0$. At the prime 7 it has bad reduction. The point $(7, 21) \in E(\mathbb{Q})$ extends uniquely to the point $(0, 0) \in \overline{E}(\mathbb{F}_7)$, which is singular.

Let $K$ be a number field (e.g. $\mathbb{Q}$ and $D$ be a Dedekind domain inside $K$, such that $\mathrm{Frac}(D) = K$ (e.g. $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$).

Given an elliptic curve $E/K$, it is not always possible to expand it to an abelian scheme over $D$. For a long time, people thought that it was in general impossible to get a model of $E$ over $D$ that has a group structure and where points can be reduced modulo primes, until André Néron came up with his construction of the Néron model.

Néron relaxed the criterion of properness and came up with the following definition.

**Definition 1.** Let $E$ be an elliptic curve over $K$. Then a Néron model $\mathcal{N}$ of $E$ over $D$ is a smooth, separated scheme over $D$ together with a $K$-isomorphism $\mathcal{N}_K \cong E$ satisfying the following universal property (*the Néron mapping property*): for any smooth sepatated scheme $X$ over $D$ any $K$-

morphism $X_K \to E$ can be uniquely extended to a $D$-morphism $X \to \mathcal{N}$.

$$\begin{array}{ccccc} X & \overset{\exists!}{\dashrightarrow} & \mathcal{N} & \longrightarrow & \operatorname{Spec} D \\ \uparrow & & \uparrow & & \uparrow \\ X_K & \longrightarrow & E & \longrightarrow & \operatorname{Spec} K \end{array}$$

By its very definition, if a Néron model exists, it is unique up to unique isomorphism. The definition via the universal property makes it so that $\mathcal{N}$ inherits from $E$ a unique structure of group scheme. To see it, one just expands the multiplication and inverse maps to $\mathcal{N} \times \mathcal{N}$ and $\mathcal{N}$ respectively. By the uniqueness part of the Néron mapping property, all properties of the group structure are preserved.

Now go back one more time to the definition and use the identity map $D \to D$ for the map $X \to D$. We find a bijection

$$E(K) \to \mathcal{N}(D).$$

On the other hand, if we have a point in $\mathcal{N}(D)$, i.e. a point with coordinates in $D$, we can take the reduction modulo a prime $\mathfrak{p}$. In this way, we obtain the desired reduction of points $E(K) \to \mathcal{N}(k(\mathfrak{p}))$.

**Theorem 1.** *Let $E$ be an elliptic curve over a number field $K$ and $D$ a Dedekind domain contained in $K$. Then $E$ admits a Néron model over $D$.*

Moreover, it turns out that if $E$ has good reduction at the primes of $D$, its minimal Weierstrass model (if it exists) is the Néron model. In the rest of the presentation we will give an idea of how Néron models look like at primes of bad reduction.

**Remark 1.** The formation of Néron models does not in general behave well under base change, but it does commute with étale base change.

**Remark 2.** Néron models can be defined more in general for abelian varieties over $K$ and even more in general for schemes. Theorem 1 is actually valid also for abelian varieties. There are instead examples of very simple schemes which do not admit Néron models:

- For the projective space $\mathbb{P}^1_{\mathbb{Q}}$, the scheme $\mathbb{P}^1_{\mathbb{Z}}$ is not a Néron model, as the automorphism of $\mathbb{P}^1_{\mathbb{Q}}$ given by $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{Q})$ does not extend to an automorphism of $\mathbb{P}^1_{\mathbb{Z}}$ in $\mathrm{PGL}_2(\mathbb{Z})$. In fact, $\mathbb{P}^1_{\mathbb{Q}}$ does not have a Néron model at all.

- As $\mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q} \neq \mathbb{Z} = \mathbb{A}^1_{\mathbb{Z}}(\mathbb{Z})$, we see that $\mathbb{A}^1_{\mathbb{Z}}$ is not a Néron model of $\mathbb{A}^1_{\mathbb{Q}}$. In fact, $\mathbb{A}^1_{\mathbb{Q}}$ does not have a Néron model.
- In the same way $\mathbb{G}_{m,\mathbb{Z}}$ is not a Néron model of $\mathbb{G}_{m,\mathbb{Q}}$. However, $\mathbb{G}_{m,\mathbb{Q}}$ does have a Néron model, but it is much more complicated.

## Néron models of elliptic curves

Here we will show how to construct a Néron model for a certain elliptic curve. Since we are going to do geometry, we will fix $R$ to be the ring of power series in one variable $\mathbb{C}[[t]]$, which is a complete discrete valuation ring. Its fraction field $K$ is $\mathbb{C}((t))$. In fact we could work with any discrete valuation ring $R$, so if you are used to the number-theoretic setting, you may think of $\mathbb{Z}_p$ and $\mathbb{Z}_{(p)}$, and their fraction fields $\mathbb{Q}_p$ and $\mathbb{Q}$. For concreteness we will work with an example of elliptic curve with split multiplicative reduction.

Let $E_1$ be the elliptic curve over $K$ defined by $y^2 = x^3 + x^2 + t$. The same equation provides a Weierstrass model $\mathcal{E}_1$ over $R$, which has bad reduction. Indeed, modulo $t$, we have a singular point. Let's remove the singular point $(x = 0, y = 0, t = 0)$ from $\mathcal{E}_1$. As previously seen, we have a group structure on the resulting model $\mathcal{E}_1^{sm}$. Moreover, the bijection

$$E_1(K) \to \mathcal{E}_1(R)$$

does factor through $\mathcal{E}_1^{sm}(R)$. Indeed, suppose by contradiction that we have a solution with $(x, y)$ with $x, y \in R$ that reduces mod $t$ to the point $(0, 0)$, that is $x \equiv 0 \bmod t$ and $y \equiv 0 \bmod t$. In the equality $y^2 = x^3 + x^2 + t$ the left-hand side is congruent to 0 mod $t^2$ and the right-hand side is congruent to $t$ mod $t^2$, which is a contradiction. In fact, it turns out that $\mathcal{E}_1^{sm}$ is the Néron model of $E_1$.

Let $E$ be the elliptic curve over $K$ defined by $y^2 = x^3 + x^2 + t^2$. The same equation gives its model $\mathcal{E}$ over $R$. The valuative criterion for properness tells us that $\mathcal{E}(R) = E(K)$. We can interpret $\mathcal{E}$ as a family of curves: for every value of the parameter $t$ we obtain a curve. For values of $t$ different from zero, we obtain an elliptic curve; for $t = 0$ we obtain a singular curve with a node $P$ (with coordinates $x = 0, y = 0, t = 0$). The ring $R = \mathbb{C}[[t]]$ should be thought of as the ring of regular functions of a small neighbourhood of the origin in $\mathbb{A}^1_{\mathbb{C}}$; we can picture $\mathcal{E}$ as a surface with a map $\pi : \mathcal{E} \to U$; the curver $E$ over $K$ corresponds to the fibre over $\{t \neq 0\}$. The fibre over $t = 0$ is the reduction of $\mathcal{E}$ mod $t$, which is singular. Our aim is to construct the Neron model over $U$ of the fibre $E$.

A first candidate for a Néron model could be the smooth locus $\mathcal{E}^{sm} = \mathcal{E} \backslash \{P\}$. Let's check whether we have $E(K) = \mathcal{E}(R) = \mathcal{E}^{sm}(R)$. Consider the $R$-valued point $s$ of $\mathcal{E}$ given by $(x = 0, y = t)$. This reduces to the singular point mod $t$. The point $s$ can be interpreted as a section $s : U \to \mathcal{E}$ going through $P$, so it does not lie in $\mathcal{E}^{sm}$. Hence in this case $\mathcal{E}(R) \neq \mathcal{E}^{sm}(R)$, and it follows that $\mathcal{E}^{sm}$ is not the Néron model of $E$.

Our next step is to introduce the procedure of blowing-up.

**Definition 2.** Let $X$ be a scheme and $Z \subset X$ a closed subscheme. The blowing-up of $X$ at $Z$ is a proper morphism $\widetilde{X} \to X$ such that $f^{-1}(Z)$ is a Cartier divisor in $X$ and satisfying the universal property: every morphism of schemes $g : Y \to X$ such that $g^{-1}(Z)$ is a Cartier divisor factors uniquely via $f$.

**Remark 3.** When the closed subscheme blown-up consists of a point $P$, the blowing-up $\widetilde{X} \to X$ looks as follows: the open $\widetilde{X} \setminus f^{-1}(P)$ is mapped by $f$ isomorphically to $X \setminus P$. The subscheme $f^{-1}(P)$ parametrizes all directions tangent to $P$ on $X$.

Now we perform the procedure of blowing-up to the singular point $P = (x = 0, y = 0, t = 0)$ inside $\mathcal{E}$ to obtain a new model $\mathcal{E}'$ of $E$, with a map $f : \mathcal{E}' \to \mathcal{E}$. We have $f^{-1}(P) \cong \mathbb{P}^1_{\mathbb{C}}$. This preimage parametrizes all directions tangent to $P$ in $\mathcal{E}$ and it contains two non-smooth points where $f^{-1}(P)$ meets the rest of the fibre over $t = 0$. The lifting $s'$ of the section $s$ to $\mathcal{E}'$ meets $f^{-1}(P)$ at the point corresponding to the direction with which $s$ goes through $P$. As $s$ meets the special fibre transversally, the lifting $s'$ lies in the smooth locus of $\mathcal{E}'$.

It turns out that $\mathcal{E}'$ satisfies $\mathcal{E}'(R) = \mathcal{E}'^{sm}(R)$; namely, no sections of $\mathcal{E}'$ go through any of the two non-smooth points. Hence we do have the equality $E(K) = \mathcal{E}'(K) = \mathcal{E}'(R) = \mathcal{E}'^{sm}(R)$ (the first equality coming from the fact that blowing-up only alters the fibre over $t = 0$). It turns out that $\mathcal{E}'^{sm}$ is indeed the Néron model of $E$. The reduction modulo $t$ of $\mathcal{E}'^{sm}$ has *two* components, both isomorphic to the multiplicative group $\mathbb{G}_m$.

More in general, given a discrete valuation ring $R$ with uniformizer $t$ and fraction field $K$ and an elliptic curve $E$ over a field $K$, we have different possibilities for the reduction modulo $t$ of its Néron model over $R$. Denoting such reduction by $\mathcal{N}_0$, there are four main cases:

- if $E$ has good reduction, $\mathcal{N}_0 \cong \mathcal{E}_0$ where $\mathcal{E}$ is the minimal Weierstrass model of $E$ over $R$. This is an elliptic curve over $R/tR$;
- if $E$ has split multiplicative reduction, $\mathcal{N}_0 \cong \mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 0$;
- if $E$ has non-split multiplicative reduction, $\mathcal{N}_0$ becomes isomorphic to $\mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$ after a quadratic extension of the base field $K \to K'$. Looking at the action of Galois we see that $N_0$ is the disjoint union of copies of $\mathbb{G}_{m,K'}$ and of some non-split tori;
- if $E$ has additive reduction, $\mathcal{N}_0 \cong \mathbb{A}^1 \times \Phi$ where $\Phi$ is a group of order at most 4.

**Remark 4.** In general, the construction of the Néron model of an elliptic curve goes via the so-called minimal regular model, which can be obtained from any model after a series of blow-ups and blow-downs. The smooth locus of the minimal regular model is the Néron model.

# Other applications

Néron models are used in work related to Mazur's theorem for elliptic curves, the Birch-Swinnerton-Dyer conjecture, the theory of heights in diophantine geometry, and in the comparison theorem, relating Néron models to jacobians of curves.