

# Duality of abelian varieties

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**Abstract.** In this talk the dual of an abelian variety will be defined. It will be shown that this dual is an abelian variety again. Using dual morphisms, we will finally show that the dual of the dual is canonically isomorphic to the original abelian variety.

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## 0 Conventions and notation

The letters  $A$  and  $B$  will be used to denote an abelian variety over a base scheme,  $S = \text{Spec } k$ , the spectrum of a field  $k$ .

## 1 Dual abelian variety

Last week, Smit discussed the Picard functor. In our case he showed that  $\text{Pic}_{A/S}$  is a separated scheme, locally of finite type over  $S$ . It represents the *relative Picard functor*

$$\text{Sch}/S \rightarrow \text{Set} : T \mapsto \text{Pic}(A_T) / \text{pr}_T^* \text{Pic}(T).$$

By construction it is a group scheme, whose identity point  $e \in \text{Pic}_{A/S}(S)$  is given by the structure sheaf  $\mathcal{O}_A$ .

**Definition 1.** *The dual of  $A$  is*

$$A^\vee := \text{Pic}_{A/S}^0,$$

the connected component of  $\text{Pic}_{A/S}$  containing the identity  $e$ .

**Example 2.** For an elliptic curve over  $S$ , i.e. an abelian variety of dimension 1, the dual is canonically isomorphic to itself, using for example Riemann-Roch.

In the rest of this section, there will be an outline of a proof of the fact that  $A^\vee$  is again an abelian variety.

**Lemma 3.** *Let  $X$  be a (proper) variety over  $S$ . Then  $\text{Pic}_{X/S}^0$  is a group scheme. In particular, the dual  $A^\vee$  is a group scheme over  $k$ .*

*Proof.* Let  $x \in \text{Pic}_{X/S}^0$  be a point and consider the translation homomorphism  $t_x : \text{Pic}_{X/S}^0 \rightarrow \text{Pic}_{X/S}$  by  $x$ . It must map the connected scheme  $\text{Pic}_{X/S}^0$  into one of the connected components of  $\text{Pic}_{X/S}$ . As  $e$  is mapped to  $x \in \text{Pic}_{X/S}^0$ , the translation by  $x$  maps  $\text{Pic}_{X/S}^0$  into  $\text{Pic}_{X/S}^0$ .  $\square$

**Lemma 4** ([BLR, Thm. 3, ch. 8.4, p. 232]). *Let  $X$  be a smooth proper variety over  $k$ . Then  $\text{Pic}_{X/S}^0$  is proper over  $k$ . In particular, the dual  $A^\vee$  of  $A$  is proper over  $k$ .*

*Proof.* We already know that  $\text{Pic}_{X/S}^0$  is separated and locally of finite type. As it is a connected group scheme, this implies that it is also of finite type. The strategy to prove this is as follows (cf. [EGM, Prop. 3.18, ch. III, p. 38]).

The connected components of  $(\text{Pic}_{X/S}^0)_{\bar{k}}$  map surjectively onto  $\text{Pic}_{X/S}^0$ . Hence, the rational point  $e$  is contained in all of them and  $(\text{Pic}_{X/S}^0)_{\bar{k}}$  is connected. Then one can prove, using general theory on group schemes, that  $(\text{Pic}_{X/S}^0)_{\bar{k}, \text{red}}$  is a smooth group scheme, implying that  $(\text{Pic}_{X/S}^0)_{\bar{k}}$  is irreducible. To get that  $\text{Pic}_{X/S}^0$  is quasi-compact, take a (dense) affine open  $U$ . For each  $x \in \text{Pic}_{X/S}^0$  the dense opens  $U$  and  $x^{-1}U$  intersect. Hence  $U \times U \rightarrow \text{Pic}_{X/S}^0 : (u, v) \mapsto u \cdot v^{-1}$  is surjective and  $\text{Pic}_{X/S}^0$  is quasi-compact. In particular, it is of finite type.

For the properness, we use the valuative criterium. Let  $R$  be a  $k$ -algebra, which is also a discrete valuation ring with field of fractions  $K$ . Then we need to show that line bundles on  $X_K$  extend uniquely to  $X_R$ . Since  $X/k$  is smooth, the procedure can be done using Weil divisors (formal sums of closed integral codimension 1 subschemes). Basically the idea is that given a divisor on  $X_K$ , you take its closure inside  $X_R$ . Details for doing this can be found in [BLR, loc. cit.] or [EGM, Cor. 6.6, Ch. VI, p. 90–91].  $\square$

Now we are only left to show that  $A^\vee$  is reduced. In order to do this, we first try to get a better understanding of the tangent space of  $A^\vee$  at  $e$ .

**Proposition 5** ([EGM, Cor. 6.6, Ch. VI, p. 90–91]). *Let  $X$  be a proper variety over  $S$ . The tangent space of  $\text{Pic}_{X/S}$  at  $e$  is (canonically) isomorphic to  $H^1(X, \mathcal{O}_X)$ .*

*Proof.* Consider  $T = \text{Spec}(k[\varepsilon]/\varepsilon^2)$ . Then intuitively the tangent space at  $e$  consists of line bundles over  $X_T$  that reduce to  $e$  modulo  $\varepsilon$  (for details consult [EGM, loc. cit.]). We have an exact sequence of sheafs

$$0 \rightarrow \mathcal{O}_X \xrightarrow{h} \mathcal{O}_{X_T}^* \rightarrow \mathcal{O}_X^* \rightarrow 1,$$

where the map  $H$  is given by  $f \mapsto 1 + \varepsilon f$ . On degree 0, the associated short sequence on cohomology is exact. Hence, on degree 1, we get the exact sequence

$$0 \rightarrow H^1(X, \mathcal{O}_X) \rightarrow \text{Pic}(X_T) \rightarrow \text{Pic}(X),$$

which is what we wanted to prove.  $\square$

**Remark 6.** Let  $C$  be a smooth proper curve over  $k$ . Then we can prove that  $\text{Pic}_{C/k}^0$  is smooth using the formal smoothness criterium. Extending the field, if necessary we may assume that  $C$  has a rational point. Then the question is whether any line bundle on  $C_{k[t]/(t^n)}$  can be lifted to  $C_{k[t]/(t^{n+1})}$ . However, the obstruction for lifting line bundles, lies in  $H^2(C, \mathcal{O}_C)$  (see for example [Har, Ch. 10]), which is 0 as  $C$  is 1-dimensional. Hence  $\text{Pic}_{C/k}^0$  is an abelian variety, which is also called the *Jacobian of  $C$* .

The following theorem uses quite a bit of theory on bialgebras. No proof will be given.

**Theorem 7** ([EGM, Prop. 6.16, ch. VI, p. 94]). *Let  $X$  be a group scheme over  $S$ . Then  $\dim_k(H^1(X, \mathcal{O}_X)) \leq \dim(X)$ .*

To continue, let us recall the universal property for the dual. We had a rigidified bundle  $\mathcal{P}$  on  $X \times \text{Pic}_{X/S}$ , the so-called *Poincaré bundle*, which satisfies the following universal property: if  $L$  is a rigidified line bundle on  $X_T$ , then there is a unique map  $\tau : T \rightarrow \text{Pic}_{X/S}$  such that  $(\text{id}_X \times \tau)^*\mathcal{P} \cong L$  (as rigidified line bundle). This universal property also exists for  $\text{Pic}_{X/S}$  if we assume the line bundle to have degree 0 everywhere.

Now let  $L$  be an ample line bundle on  $A$ . Recall how we defined the *Mumford bundle*

$$\Lambda(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$$

on  $A \times A$ . We can put its canonical rigidification, consider  $\Lambda(L)$  as family of line bundles on  $A \times \{e\}$  parametrised over  $\{e\} \times A$ , and find an associated morphism

$$\varphi_L : A \rightarrow \text{Pic}_{A/S},$$

such that  $(\text{id}_A \times \varphi_L)^*\mathcal{P} \cong \Lambda(L)$ . This morphism is given by  $x \mapsto t_x^*L \otimes L^{-1}$  and as  $A$  is connected, it factors through  $A^\vee$ .

**Theorem 8.** *The kernel  $K(L)$  of  $\varphi_L$  is finite. Therefore,*

$$\dim(A^\vee) = \dim_k H^1(X, \mathcal{O}_X).$$

*In particular,  $A^\vee$  is smooth.*

*Proof.* This kernel  $K(L)$  has been studied extensively before, in the talk by Bindt and/or Winter. Recall that we showed that it is finite, if  $L$  is ample. Hence,  $\varphi_L$  is finite and  $\dim(A) = \dim(A^\vee)$ . We get the other inequality  $\dim(A^\vee) \geq \dim_k H^1(X, \mathcal{O}_X)$ . Together with the one we already had, we get the desired statement about the dimensions. Hence,  $A^\vee$  is smooth at 0 and by translation it is smooth everywhere.  $\square$

**Corollary 9.** *In particular,  $A^\vee$  is an abelian variety.*

## 2 Dual morphisms

Given a morphism between two abelian varieties, we can define its dual as follows.

**Definition 10.** Let  $f : A \rightarrow B$  be a morphism of abelian varieties. Then its *dual*

$$f^\vee : B^\vee \rightarrow A^\vee$$

is defined by taking the pullback on line bundles.

**Remark 11.** The reason that this is well-defined is as follows. From this construction, in principle, we only get a map  $B^\vee \rightarrow \text{Pic}_{A/S}$ . However, as  $B^\vee$  is connected, the image must be connected. Hence, it lies in the connected component of the zero section, which is  $A^\vee$ .

In this set-up, we can dualise kernels of isogenies in the following sense.

**Theorem 12** ([Milne, Thm. 9, p. 41]). *Let  $f : A \rightarrow B$  be an isogeny with kernel  $N$  (as group scheme). Then  $f^\vee$  is an isogeny with kernel  $N^\vee$ , the Cartier dual of  $N$ , the finite group scheme which is given by*

$$\underline{\text{Hom}}(-, \mathbb{G}_m) : T \mapsto \text{Hom}_{\text{GrpSch}/T}(G_T, \mathbb{G}_{m,T}).$$

Here follow some useful observations that may help you to understand this Cartier dual better.

**Remark 13.** The rank of  $N^\vee$  equals the rank of  $N$ .

**Remark 14.** On the level of Hopf algebras, the Cartier dual is given on the dual vector space by swapping the multiplication and the comultiplication.

**Example 15.** The Cartier dual of the constant group scheme  $\mathbb{Z}/n\mathbb{Z}$  is the group scheme  $\mu_n = \text{Spec}(k[x]/(x^n - 1))$ . The Cartier dual of  $\ker(\text{Frob} : S \rightarrow S)$  (for  $S$  of characteristic  $p$ ) is itself.

In order to prove Theorem 12, we consider extensions of  $A$  by  $\mathbb{G}_m$ .

**Lemma 16.** *Let  $L$  be a line bundle on  $A$  of degree 0, rigidified along the section  $e : S \rightarrow A$ , i.e. a line bundle  $L$  together with an isomorphism  $e^*L \cong \mathcal{O}_S$ . Then  $L$  corresponds to an extension (of group schemes)*

$$\mathbb{G}_{m,S} \hookrightarrow E \twoheadrightarrow A.$$

*In particular, the sheaf  $\underline{\text{Ext}}^1(A, \mathbb{G}_m)$  is canonically represented by  $A^\vee$ .*

*Proof outline.* Let  $E$  be the sheaf  $\underline{\text{Isom}}(\mathcal{O}_A, L)$  on  $\text{Sch}/A$ . Then  $E$  is representable by a scheme over  $A$ , which we will also denote by  $E$  (cf. [FGA, Thm. 5.8, ch. 5, p. 120]). Now, we consider  $E$  as scheme over  $S$ , by composition with the structure map  $A \rightarrow S$ . For an  $S$ -scheme  $T$  consider two points  $(a, \alpha), (b, \beta) \in E(T)$ . The data in a point consists of a section  $a \in A(T)$  (resp.

$b \in A(T)$ ) and an isomorphism  $\alpha : \mathcal{O}_T \rightarrow a^*L$  (resp.  $\beta : \mathcal{O}_T \rightarrow b^*L$ ). Now the theorem of the square gives an isomorphism

$$(a + b)^*L \cong a^*L \otimes b^*L \otimes (e_T^*L)^{-1}.$$

As the line bundle is rigidified along  $e$ , we have an isomorphism  $e_T^*L \cong \mathcal{O}_T$ . Hence we get an isomorphism  $(a + b)^*L \cong \mathcal{O}_T$  for free, which we will call  $\alpha + \beta$  and which defines the addition in  $E$ .

There is a map of  $\mathbb{G}_{m,S}$  into  $E$ , mapping  $x \in \mathbb{G}_{m,S}(T)$  to the pair  $(e_T, \chi)$ , where  $\chi$  is the isomorphism  $\mathcal{O}_T \rightarrow e_T^*L$  given by multiplication with  $x$  (when you identify  $e_T^*L$  with  $\mathcal{O}_T$  using the rigidification). Moreover, there is also an obvious map from  $E$  to  $A$  (forgetting the isomorphism). It is not hard to check that  $E$  is a torsor over  $A$  for the action of  $\mathbb{G}_{m,S}$  and this sequence is exact.  $\square$

*Proof for theorem 12.* Consider the (contravariant) functor  $\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ . By applying it to the sequence  $0 \rightarrow N \rightarrow A \rightarrow B \rightarrow 0$ , we get the following sequence from the long exact sequence:

$$0 \rightarrow \underline{\mathrm{Hom}}(N, \mathbb{G}_m) \rightarrow \underline{\mathrm{Ext}}^1(B, \mathbb{G}_m) \rightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) \rightarrow 0.$$

Here we remark that  $\underline{\mathrm{Hom}}(A, \mathbb{G}_m) = 0$  as  $A$  is projective and  $\underline{\mathrm{Ext}}^1(N, \mathbb{G}_m) = 0$  as  $N$  is 0-dimensional. In the previous lemma we saw how  $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$  and  $\underline{\mathrm{Ext}}^1(B, \mathbb{G}_m)$  are identified with  $A^\vee$  and  $B^\vee$ , which finishes the proof.  $\square$

**Example 17.** Let  $E/S$  be an elliptic curve and  $\varphi : E \rightarrow E$  the multiplication by  $n \in \mathbb{Z}$ . Then  $E$  is canonically isomorphic to its own dual and the dual morphism  $\varphi^\vee$  of  $\varphi$  is  $\varphi$  itself. Hence, we get an isomorphism, for the kernels of the isogeny and its dual:  $E[n] = \ker(\varphi) \cong \mathrm{Hom}(\ker(\varphi^\vee), \mathbb{G}_m) = \mathrm{Hom}(E[n], \mathbb{G}_m)$ . This gives rise to a bilinear map  $E[n] \times E[n] \rightarrow \mu_n \subset \mathbb{G}_m$ , which is more commonly known as the *Weil pairing* on  $E$ .

### 3 Dual of the dual

Now we can consider the identity map  $A^\vee \rightarrow A^\vee$  which corresponds to the Poincaré bundle  $\mathcal{P}$  on  $A \times A^\vee$ . If we now consider this line bundle as a family of line bundles on  $A^\vee$  parametrised by  $A$ , we get a canonical morphism  $\kappa_A : A \rightarrow (A^\vee)^\vee$ . Our goal will be to prove that this is an isomorphism, justifying why  $A^\vee$  is called the dual of  $A$ . For this we need the following proposition.

**Proposition 18.** *Let  $L$  be an ample line bundle on  $A$  and  $\varphi_L : A \rightarrow A^\vee$  the associated morphism. Then the composition*

$$A \xrightarrow{\kappa_A} (A^\vee)^\vee \xrightarrow{\varphi_L^\vee} A^\vee$$

*is  $\varphi_L$ .*

*Proof.* Let  $T/S$  be any scheme and  $x \in A(T)$  any element. We have the following commutative diagram.

$$\begin{array}{ccc} T \times A^\vee & \xrightarrow{x \times \text{id}_{A^\vee}} & A \times A^\vee \\ \text{id}_T \times \varphi_L \uparrow & & \uparrow \text{id}_A \times \varphi_L \\ T \times A & \xrightarrow{x \times \text{id}_A} & A \times A \end{array}$$

If we start in the upper right corner with the Poincaré bundle  $\mathcal{P}$ , then, on the one hand,  $(x \times \text{id}_{A^\vee})^* \mathcal{P}$  is  $\kappa_A(x)$ . Moreover,  $(\text{id}_T \times \varphi_L)^* \kappa_A(x)$  is  $\varphi_L^\vee(\kappa_A(x))$ . To summarise,

$$\varphi_L^\vee(\kappa_A(x)) \cong (x \times \varphi_L)^* \mathcal{P}.$$

On the other hand, we have  $(\text{id}_A \times \varphi_L)^* \mathcal{P} \cong \Lambda(L)$ . You can see this, as on points  $y$  the map  $\varphi_L$  is given by  $y \mapsto t_y^* L \otimes L^{-1}$ , and the symmetric sheaf

$$\Lambda(L) = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

also reduces to this sheaf on the points  $y$  (note that  $p_2^* L^{-1}$  is trivial in this Picard group). Then we can argue in the same way to see that

$$(x \times \text{id}_A)^* \Lambda(L) \cong t_x^* L \otimes L^{-1} \cong \varphi_L(x). \quad \square$$

**Corollary 19.** *The morphism  $\kappa_A : A \rightarrow (A^\vee)^\vee$  is an isomorphism.*

*Proof.* By the previous proposition, we immediately see that it is finite. Moreover, as the degree of  $\varphi_L$  and  $\varphi_L^\vee$  are equal (as Cartier dual finite group schemes have equal rank),  $\kappa_A$  has degree 1.  $\square$

## References

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