Models of curves

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These are notes for a talk given in the Néron model seminar held in Leiden, The Netherlands, in the fall of 2017. All readers are encouraged to e-mail r.van.bommel@math.leidenuniv.nl in case of errors, unclarities and other imperfections. Most of this talk is based on chapters 8, 9 and 10 of [Liu02].

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Throughout this talk, let S be a Dedekind scheme of dimension 1, η its generic point and s a closed point. Let k be the residue field of s and K be the function field of S. Let C/K be a smooth projective curve of genus g.

1 Regular models

Definition 1. A *fibred surface* X/S is an integral, projective, flat S-scheme of relative dimension 1. If X is regular/normal, then it is called a *regular/normal fibred surface.*

Definition 2. A *model* $\mathcal{C}/S$ of C is a normal fibred surface together with an isomorphism $\mathcal{C}_\eta \cong C$. If $\mathcal{C}$ is regular, the model is called a *regular model.*

Example 3. Consider the elliptic curve $E : Y^2 = X^3 + 1$ over $\mathbb{Q}$. Then $\mathcal{E} = Z(y^2z - x^3 - z^3) \subset \mathbb{P}^2_{\mathbb{Z}}(x : y : z)$ is a model of E in the obvious way.

The locus where $\mathcal{E}/\mathbb{Z}$ is not smooth, is the locus where $-3x^2$, $2yz$ and $y^2 - 3z^2$ are zero (Jacobian criterion). Over characteristic unequal to 2 or 3, this does not happen, as it would imply that all three of x, y and z are zero. In $\mathcal{E}_{\mathbb{F}_2}$, the point $(0 : -1 : 1)$ is the only non-smooth point, and in $\mathcal{E}_{\mathbb{F}_3}$ the point $(-1 : 0 : 1)$ is the only non-smooth point.

In order to check for regularity, we remark that

$$y^2 - x^3 - 1 = (y+1)^2 - x^3 - 2(y+1) \in (2, x, y+1)^2 \subset \mathbb{Z}[x, y],$$

and in the latter case, we get

$$y^2 - x^3 - 1 = y^2 - (x+1)^3 - 3x(x+1) \in (3, x+1, y)^2 \subset \mathbb{Z}[x,y].$$

In both cases the equation lies in $\mathfrak{m}^2$, and will not give rise a an equation in $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal corresponding to the point. Therefore, both non-smooth points turn out to be non-regular, and $\mathcal{E}$ is not a regular model.

Problem 4. Show that $\mathcal{E} : Z(y^2z - x^3 - xz^2) \subset \mathbb{P}^2_{\mathbb{Z}}(x : y : z)$ is a regular model of the elliptic curve given by $Y^2 = X^3 + X$.

In the next sections, we will show that in very reasonable circumstances, regular models of curves exist and can be constructed in practice.

2 Blowing-ups

Definition 5. Let A be a noetherian ring and let I be an ideal of A. Let $\widetilde{A}$ be the graded A-algebra $\bigoplus_{d\geq 0} I^d$. Then the *blowing-up of* $X = \operatorname{Spec} A$ *along* $V(I)$ is defined as the natural morphism

$$\widetilde{X} = \operatorname{Proj} \widetilde{A} \to X.$$

Cf. [Liu02, Lemma 8.1.8, p. 321], these blowing-ups of affine schemes can be glued. This will allow us to define a blowing up of a general scheme along the zero locus of a coherent sheaf of ideals.

Example 6. Let $A = \mathbb{Z}[x,y]/(y^2 - (x-1)^3 - 1)$. Then $X = \operatorname{Spec} A$ has a so-called cuspidal singularity in the point $(0,0)$ in the fibre above 3 (see also Example 3). Let $I = (x, y, 3)$, and consider the blowing-up $\widetilde{X} \to X$ of X along $V(I)$.

We let T_1, T_2 and T_3 be the elements x, y and 3 in the degree 1 part of $\widetilde{A}$ (not to be confused with the elements in the degree 0 part). Then $\widetilde{X}$ is covered by $D_+(T_1), D_+(T_2)$ and $D_+(T_3)$.

Then, according to [Liu02, Lemma 8.1.4, p. 320] we have

$$D_+(T_1) = \operatorname{Spec} A_1, \quad D_+(T_2) = \operatorname{Spec} A_2, \quad D_+(T_3) = \operatorname{Spec} A_3,$$

where A_1 is the sub-A-algebra of $A[1/x]$ generated by $\frac{y}{x}$ and $\frac{3}{x}$, A_2 is the sub-A-algebra of $A[1/y]$ generated by $\frac{x}{y}$ and $\frac{3}{y}$ and A_3 is the sub-A-algebra of $A[1/3]$ generated by $\frac{x}{3}$ and $\frac{y}{3}$.

One would be inclined to take

$$\begin{aligned} B_1 &= \mathbb{Z}[x, y, w_1, z_1]/(y^2 - (x-1)^3 - 1, w_1 x - y, z_1 x - 3) \\ &= \mathbb{Z}[x, w_1, z_1]/(w_1^2 x^2 - (x-1)^3 - 1, z_1 x - 3), \end{aligned}$$

but be aware that in A_1, there are an extra relations, for example

$$yw_1 = \frac{y^2}{x} = \frac{x^3 - 3x^2 + 3x}{x} = x^2 - 3x + 3,$$

or in other words, $w_1^2 x = x^2 - 3x + 3$ (which does not hold in B_1). Also define

$$\begin{aligned} B_2 &= \mathbb{Z}[x, y, w_2, z_2]/(y^2 - (x-1)^3 - 1, w_2 y - x, z_2 y - 3) \\ &= \mathbb{Z}[y, w_2, z_2]/(y^2 - (w_2 y - 1)^3 - 1, z_2 y - 3) \end{aligned}$$

and

$$\begin{aligned} B_3 &= \mathbb{Z}[x, y, w_3, z_3]/(y^2 - (x-1)^3 - 1, 3w_2 - x, 3z_3 - y) \\ &= \mathbb{Z}[w_3, z_3]/(9z_3^2 - (3w_3 - 1)^3 - 1). \end{aligned}$$

The advantage of using these rings is that they can be computed with a computer. The schemes $\operatorname{Spec} B_1, \operatorname{Spec} B_2$ and $\operatorname{Spec} B_3$ glue to a scheme B that is the inverse image of X in the blow-up of $\mathbb{A}^2_{\mathbb{Z}}(x, y)$ along $V(I)$.

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \widetilde{\mathbb{A}^2_{\mathbb{Z}}} \arrow[d, "\varphi\text{, blow-up along } V(I)"] \\ X \arrow[r] & \mathbb{A}^2_{\mathbb{Z}} \end{tikzcd}$$

The problem is that B contains too much. It contains the whole exceptional fibre $E \subset \widetilde{\mathbb{A}^2_{\mathbb{Z}}}$. In fact, one can obtain $\widetilde{X}$ be taking the *strict transform* of X inside $\widetilde{\mathbb{A}^2_{\mathbb{Z}}}$, i.e. the closure of $\varphi^{-1}(X \setminus V(I))$ inside $\widetilde{\mathbb{A}^2_{\mathbb{Z}}}$. Without providing any proof or argument, I will explain you how to do this. To do this, the only thing we need is that X is not contained in $V(I)$, see [Liu02, Cor. 8.1.17, p. 324]

The equation $z_1 x - 3$ in B_1 comes from the blowing-up of $\mathbb{A}^2_{\mathbb{Z}}$ in $V(I)$, so that one we do not change. In B_1 the exceptional fibre is defined by $x = 0$, so we will try to get rid of some factors x in the first equation. For the original equation $y^2 - (x-1)^3 - 1$, we already saw that it lies in I^2, but not in I^3. Therefore, the first equation $w_1^2 x^2 - (x-1)^3 - 1$ is divisible by x^2. This is not immediately obvious, but we can rewrite it as

$$w_1^2 x^2 - x^3 + 3x^2 - 3x = x^2(w_1^2 - x + 3) - z_1 x^2.$$

Hence, we now consider

$$\begin{aligned} C_1 &= \mathbb{Z}[x, w_1, z_1]/(w_1^2 - x + 3 - z_1, z_1 x - 3) \\ &= \mathbb{Z}[x, w_1]/((w_1^2 - x + 3)x - 3) \end{aligned}$$

and, analogously,

$$C_2 = \mathbb{Z}[y, w_2, z_2]/(1 - w_2^3 y + 3w_2^2 - w_2 z_2, z_2 y - 3)$$

and

$$C_3 = \mathbb{Z}[w_3, z_3]/(z_3^3 - 3w_3^3 + 3w_3^2 - w_3).$$

Gluing the C_i in the obvious way, we get $\widetilde{X}$. We see that the special fibre (above 3) of C_1 is the intersection of a line and a parabola. In C_2 we get the disjoint union of two copies of $\mathbb{G}_m$, and in C_3 we get one copy of $\mathbb{A}^1$.

Problem 7. Check how these C_i glue, and prove that $\widetilde{X}$ does not have non-regular points in the fibre above 3.

3 Resolution of singularities

Let X/S be a fibred surface. Let $X_1 \to X$ be the normalisation of X. For $i = 2, \ldots$, let X_i be the normalisation of the blow-up of X_{i-1} along the non-regular locus of X_{i-1}. In this way, we obtain a sequence of fibred surfaces

$$\ldots \to X_{n+1} \to X_n \to \ldots \to X_1 \to X.$$

Theorem 8 ([Liu02, Cor. 8.3.51, p. 365])**.** *Suppose that the generic fibre X_η of X is smooth over K. Then this sequence stops, i.e. for some n the fibred X_n is regular. In particular, X admits a so-called desingularisation in the strong sense.*

The proof of is theorem is beyond the scope of this talk, but I would like to stress that it is really a non-trivial result. The problem of resolution of singularities has been solved for reduced algebraic varieties over fields of characteristic 0 by Hironaka. It is, however, still open for varieties of dimension at least 4 over fields of characteristic $p > 0$.

4 Minimal regular models

Now we know how to (really) resolve singularities and create regular models of curves, we will focus on simplifying our regular models. For this, we first need the following definition.

Definition 9. Let X/S be a regular fibred surface. Then X/S is called *minimal* if every birational map of regular fibred surfaces $Y/S \dashrightarrow X/S$ is a morphism.

Definition 10. A minimal regular fibred surface X/S that is at the same time a regular model of C, is called a *minimal regular model* of C.

In order to get a minimal regular model from a regular model, one needs to contract so-called exceptional divisors.

Definition 11. Let X/S be a regular fibred surface. Then a prime divisor E is called an *exceptional divisor*, if there exists a regular fibred surface Y/S and a morphism $f : X \to Y$ over S, such that $f(E)$ is a point and f is the blow-up of Y along $f(E)$.

The following criterion due to Castelnuovo provides a practical way to determine the exceptional divisors of a regular fibred surface.

Theorem 12 ([Liu02, Thm. 9.3.8, p. 416])**.** *Let X/S be a regular fibred surface and let $E \subset X_s$ be a vertical prime divisor. Moreover, let $k' = H^0(E, \mathcal{O}_E)$. Then E is an exceptional divisor if and only if $E \cong \mathbb{P}^1_{k'}$ and $E^2 = -[k' : k(s)]$.*

The following theorem, due to Lichtenbaum and Shafarevich, states that minimal regular models exists for $g \geq 1$. One can obtain a minimal regular model by taking any regular model and contracting exceptional divisors until there are none left, cf. [Liu02, Prop. 9.3.19, p. 421]. One might wonder why this only works for curves with positive genus. The main reason for this lies within the use of intersection theory.

Theorem 13 ([Liu02, Thm. 9.3.31, p. 422])**.** *If $g \geq 1$, then there exists a minimal regular model of C. Moreover, it is uniquely unique.*

4.1 $\mathbb{P}^1$ does not have a minimal regular model

Suppose that we consider $\mathbb{P}^1_{\mathbb{Q}}$ and the regular model $X = \mathbb{P}^1_{\mathbb{Z}}$. Then X is a relatively minimal surface in the sense of [Liu02, Def. 9.3.12, p. 418], i.e. X does not have any exceptional divisors. In particular, if $\mathbb{P}^1_{\mathbb{Q}}$ has a minimal regular model $\mathcal{X}$, then X has to be a minimal regular model (as there obviously exists a birational map, and hence a morphism $X \to \mathcal{X}$, and this has to be an isomorphism as X is relatively minimal, see loc. cit.).

Proposition 14 ([Liu02, Prop. 9.3.13])**.** *If X/S is a minimal regular fibred surface, then the natural map $\rho : \mathrm{Aut}_S(X) \to \mathrm{Aut}_K(X_\eta)$ is bijective.*

Proof. Due to separatedness, ρ is injective. Due to the defining property of minimal regular fibred surfaces, any automorphism $X_\eta \to X_\eta$ extends to a morphism $X \to X$. The same holds for the inverse, hence we get an automorphism and ρ is surjective. $\square$

The map $\mathrm{PGL}_2(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Q})$ is not bijective as

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

is not in the image, for example. Hence, X is not a minimal regular model of the curve $\mathbb{P}^1_{\mathbb{Q}}$.

Remark 15. Another way to obtain the same result is the following. Take a closed point P of X in the special fibre above p, and consider the blow-up $Y \to X$ of X along P. Then the special fibre above p of Y consists of two copies of $\mathbb{P}^1$, and both of them are contractible. If you contract the newly created copy of $\mathbb{P}^1$ in the aforementioned special fibre, you obtain a contraction $Y \to Z$. Now Z is another relatively minimal model of $\mathbb{P}^1_{\mathbb{Q}}$, but it is not isomorphic (as model of $\mathbb{P}^1_{\mathbb{Q}}$!) to X.

For example, if you take the point $P = (0 : 1)$ in the fibre above 2, and you execute this procedure, then the two models you get are isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$ as scheme. However, the birational map that you would get, on the special fibre $\mathbb{P}^1_{\mathbb{Q}}$, when you consider the two schemes as models (and not just as schemes), would be given by $(x : y) \mapsto (x : 2y)$, and this does not extend to a morphism on $\mathbb{P}^1_{\mathbb{Z}}$.

Problem 16. Verify the latter example: i.e. take the scheme $\mathbb{P}^1_{\mathbb{Z}}$, blow it up in P and then contract the old divisor in the special fibre. While doing this, keep track of the structure as a model of $\mathbb{P}^1_{\mathbb{Q}}$.

Remark 17. For elliptic curves, there exists an explicit classification of the special fibres of the minimal regular model. Tate's algorithm (see [Liu02, Rem. 10.2.4, p. 489]) takes the coefficients of the equation as input, and will give an explicit description the special fibre. On the background, there is of course the whole process of blowing-up the non-regular locus.

5 Néron models

Let X/S be a minimal regular model of C. The following result has first been proven for elliptic curves, and later for general curves of positive genus by Tong and Liu.

Theorem 18 ([LT16])**.** *Suppose that $g \geq 1$. Then the (non-empty, open) smooth locus $\mathcal{N}/S$ of X/S is a Néron model of C/S.*

Corollary 19. *For curves of positive genus, Néron models exist.*

Remark 20. Again, for reasons very similar as before, $\mathbb{P}^1$ does not have a Néron model.

Example 21. In the case of Example 3, the model we found after blowing-up the point $(-1:0:1)$ in $\mathcal{E}_{\mathbb{F}_3}$, is minimal regular outside the prime 2. The special fibre above 3 of the Néron model consists of two copies of $\mathbb{A}^1$, cf. Problem 7. This corresponds to the fact that E has additive reduction at 3.

Remark 22. Suppose that we work over a discrete valuation ring and we consider an elliptic curve over K. Then there are a few cases (see also the classification due to Kodaira and Néron):

- If E has good reduction, then the special fibre of the Néron model is just the reduction of any/the smooth model.
- If E has split multiplicative reduction, the special fibre of the Néron model is $\mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$ for some n.
- If E has non-split multiplicative reduction, then after an unramified[1] quadratic extension, giving rise to the extension k'/k of residue fields, the special fibre of the Néron model becomes $\mathbb{G}_{m,k'} \times \mathbb{Z}/n\mathbb{Z}$. There is only one way the Galois group can act on the component group: the non-trivial element must act by sending an element to its inverse. The special fibre of the Néron model (over S) is the union of one or two copies (if n is odd or even, respectively) of $\ker\left(\mathbb{G}_{m,k'} \xrightarrow{N_{k'/k}} \mathbb{G}_m\right)$ and $\lfloor \frac{n-1}{2} \rfloor$ non-[geometrically irreducible] components without rational point.
- If E had additive reduction, then the component group is a group of order at most 4, and the identity component is an $\mathbb{A}^1$. In case $\operatorname{char} k > 3$, the exact sequence splits and the k-rational points of the special fibre are $\mathbb{A}^1(k) \times \Phi$, where Φ is a finite group of order at most 4.

Problem 23. Take your favourite elliptic curve with multiplicative reduction and find the special fibre of the Néron model.

References

[Liu02] Q. Liu, *Algebraic Geometry and Arithmetic Curves.* Oxford University Press, Oxford, 2002. Translated by R. Erné.

[LT16] Q. Liu, J. Tong, Néron models of algebraic curves. *Trans. Amer. Math. Soc.* **368** (2016), no. 10, 7019–7043.

[1] The Néron model is compatible with unramified extensions only!