The Grothendieck monodromy theorem

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These are notes for a talk held at the local Galois representation seminar in Leiden, The Netherlands, on Tuesday 28 April. The author apologises for all errors, unclarities, omissions of details and other imperfections and encourages the reader to send them by email to the author: r.van.bommel@math.leidenuniv.nl. The notes are mainly based on [FoOu].

1 Basic definitions

In this section the main objects and their properties will be defined.

Definition 1 (ℓ-adic representation)**.** Let K be a field and let L/K be a Galois extension with Galois group G. An *ℓ-adic representation of G* is a finite dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous and linear action of G. Moreover, if $L = K^{\text{sep}}$ such a representation is called an *ℓ-adic Galois representation.*

Example 2. Take any Galois extension with group G and any finite dimensional $\mathbb{Q}_\ell$-vector space V and let G act trivially on V, id est, $g \cdot v = v$.

There are several ways to construct ℓ-adic representations out of existing ones. Let G be as before and let r be a non-negative integer. Let V and V' be two ℓ-adic representations of G. Then the *direct sum* $V \oplus V'$, *tensor product* $V \otimes_{\mathbb{Q}_\ell} V'$, the *$r$-th symmetric power* $\operatorname{Sym}^r_{\mathbb{Q}_\ell} V$ and the *r-th exterior power* $\bigwedge^r_{\mathbb{Q}_\ell} V$ naturally carry the structure of an ℓ-adic representation. Moreover, the *dual* $V^* := \operatorname{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$ is an ℓ-adic representation by setting $(g \cdot \varphi)(v) = \varphi(g^{-1} \cdot v)$.

Definition 3 (irreducible/semisimple)**.** A representation is said to be *irreducible* if it has exactly two subrepresentations and it is said to be *semisimple* if it is a direct sum of irreducible representations.

For representations that are not semisimple, there is a way to construct a semisimple one.

Definition 4 (semisimplification)**.** Let V be a representation over some group G and let $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ be a Jordan-Hölder decomposition. Then the *semisimplification* of V is defined as $\bigoplus_{i=1}^n V_i/V_{i-1}$.

2 Examples of ℓ-adic Galois representations

Let K be a field, K^{sep} a separable closure and $G = \text{Gal}(K^{\text{sep}}/K)$. Let ℓ be a prime number not equal to the characteristic of K. Now let

$$\boldsymbol{\mu}_{\ell^n}(K^{\text{sep}}) := \{x \in K^{\text{sep}} : x^{\ell^n} = 1\}.$$

They form an inverse system by taking the following maps:

$$\boldsymbol{\mu}_{\ell^{n+1}}(K^{\text{sep}}) \to \boldsymbol{\mu}_{\ell^n}(K^{\text{sep}}) : x \mapsto x^{\ell}.$$

Definition 5 (Tate module of $\mathbb{G}_m$)**.** The limit of this system is called the *Tate module of* $\mathbb{G}_m$ or $T_\ell(\mathbb{G}_m) = \mathbb{Z}_\ell(1)$. It is a free $\mathbb{Z}_\ell$-module of rank 1 by letting $\gamma = (\gamma_n \mod \ell^n)_n \in \mathbb{Z}_\ell$ act on $t = (t_n)_n \in T_\ell(\mathbb{G}_m)$ by $\gamma \cdot t = (t_n^{\gamma_n})_n$. It has a natural action of G on it and

$$\mathbb{Q}_\ell(1) = V_\ell(\mathbb{G}_m) := \mathbb{Q}_\ell \times_{\mathbb{Z}_\ell} T_\ell(\mathbb{G}_m)$$

is an ℓ-adic representation.

Definition 6 (Tate twist)**.** For non-negative integers r we define $\mathbb{Q}_\ell(r)$ to be $\text{Sym}^r_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell(1))$ and $\mathbb{Q}_\ell(-r)$ as its dual. For any ℓ-adic representation V we define its r-th Tate twist, for $r \in \mathbb{Z}$, as

$$V(r) := V \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(r).$$

Now let A be an abelian variety of dimension g over K, i.e., a smooth projective variety with a group structure (e.g. an elliptic curve). The theory of abelian varieties, see for example [Mum], tells us that $A(K^{\text{sep}})$ is an abelian group and that its ℓ^n-torsion is isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$ and that its p^n-torsion is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$ with $0 \leqslant r \leqslant g$, if $p = \text{char}(K) \neq 0$. As before there are maps from the ℓ^{n+1}-torsion (resp. p^{n+1}) to the ℓ^n-torsion (resp. p^n) be multiplying by ℓ (resp. p).

Definition 7 (Tate module of A)**.** The limit of this system is called the *Tate module of* A or $T_\ell(A)$. It is a free $\mathbb{Z}_\ell$-module of rank $2g$ (resp. r) equipped with a natural action of G. We define the ℓ-adic representation

$$V_\ell(A) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(A).$$

Another thing we could consider are the cohomology groups, where $m \in \mathbb{N} \cup \{0\}$,

$$H^m_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_{n \in \mathbb{N}} H^m((A_{K^{\text{sep}}})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

The theory of abelian varieties tells us that $H^m_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)$ is a finite dimensional vector space. There is a natural G-action that makes them into ℓ-adic representations.

Lemma 8. *There is an isomorphism*

$$H^m_{\acute{e}t}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell) \cong \bigwedge^m_{\mathbb{Q}_\ell} (V_\ell(A))^*.$$

Proof. The theory of abelian varieties ([Mil, Th. 12.1, p. 55]) gives a canonical isomorphism $H^m_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell) = \bigwedge^m_{\mathbb{Q}_\ell} H^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)$. Hence, we only have to consider the case $m = 1$.

Consider the Kummer exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\cdot \ell^n} \mathbb{G}_m \longrightarrow 1$$

of sheaves on $A_{K^{\text{sep}}}$. The associated long exact sequence becomes

$$(K^{\text{sep}})^* \xrightarrow{\cdot \ell^n} (K^{\text{sep}})^* \to H^1(A_{K^{\text{sep}}}, \boldsymbol{\mu}_{\ell^n}) \to H^1(A_{K^{\text{sep}}}, \mathbb{G}_m) \xrightarrow{\cdot \ell^n} H^1(A_{K^{\text{sep}}}, \mathbb{G}_m)$$

As K^{sep} contains all ℓ^n-th powers, the first morphism is surjective. Hence, we find that

$$H^1(A_{K^{\text{sep}}}, \boldsymbol{\mu}_{\ell^n}) = \ker(H^1(A_{K^{\text{sep}}}, \mathbb{G}_m) \xrightarrow{\cdot \ell^n} H^1(A_{K^{\text{sep}}}, \mathbb{G}_m)) = A^\vee(K^{\text{sep}})[\ell^n],$$

as the torsion part of Pic(A) is contained in $A^\vee = \text{Pic}^0(A)$. The Weil pairing gives a canonical isomorphism $A^\vee(K)[\ell^n] = \text{Hom}(A(K)[\ell^n], \boldsymbol{\mu}_{\ell^n}(K^{\text{sep}}))$. Now we can tensor both sides with the G-module $\text{Hom}(\boldsymbol{\mu}_{\ell^n}(K^{\text{sep}}), \mathbb{Z}/\ell^n\mathbb{Z})$ to get

$$\begin{aligned} H^1(A_{K^{\text{sep}}}, \mathbb{Z}/\ell^n\mathbb{Z}) &= H^1(A_{K^{\text{sep}}}, \boldsymbol{\mu}_{\ell^n}) \otimes \text{Hom}(\boldsymbol{\mu}_{\ell^n}(K^{\text{sep}}), \mathbb{Z}/\ell^n\mathbb{Z}) \\ &= \text{Hom}(A(K)[\ell^n], \mathbb{Z}/\ell^n\mathbb{Z}). \end{aligned}$$

By taking the limit we find that $H^1(A_{K^{\text{sep}}}, \mathbb{Q}_\ell) = V_\ell(A)^*$. □

Remark 9. This construction of ℓ-adic Galois representations for the cohomology groups can be generalised if we replace $A_{K^{\text{sep}}}$ by a proper smooth variety over K^{sep}.

3 ℓ-adic representations of local fields

In this section we will suppose that K is a local field, i.e. K has a discrete valuation and is complete with respect to it, with a perfect residue field k of characteristic $p \notin \{0, \ell\}$. Let $\mathcal{O}$ be its ring of integers and $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. Let K^{sep} be a separable closure of K and $G_K = \text{Gal}(K^{\text{sep}}/K)$.

Definition 10 (inertia and wild inertia)**.** Let L/K be a finite Galois extension. Then L itself is a local field, with ring of integers $\mathcal{O}_L$ and maximal ideal $\mathfrak{m}_L$. The *inertia subgroup* $I_{L/K}$ of $\text{Gal}(L/K)$ is the subgroup of automorphisms that act trivially on $\mathcal{O}_L/\mathfrak{m}_L$. The *wild inertia subgroup* $P_{L/K}$ of $\text{Gal}(L/K)$ is the subgroup of automorphisms that act trivially on $\mathcal{O}_L/\mathfrak{m}_L^2$.

The *inertia subgroup* I_K of G_K is the limit of the inertia subgroups $I_{L/K}$ and the *wild inertia subgroup* P_K of G_K is the limit of the wild inertia subgroups $I_{L/K}$. The following proposition shows that this definition makes sense.

Proposition 11. *Let L/K and M/K be finite Galois extensions satisfying $L \subset M$. Then the image of $I_{M/K}$ and $P_{M/K}$ in* $\text{Gal}(L/K)$ *is $I_{L/K}$ resp. $P_{L/K}$.*

Proof. Let $(\mathcal{O}_K, \mathfrak{m}_K)$, $(\mathcal{O}_L, \mathfrak{m}_L)$, $(\mathcal{O}_M, \mathfrak{m}_M)$ be the rings of integers of K, L resp. M. The natural map $\mathcal{O}_L \to \mathcal{O}_M$ is a local morphism. Hence, it induces an injection $\mathcal{O}_L/\mathfrak{m}_L \hookrightarrow \mathcal{O}_M/\mathfrak{m}_M$ and every $\sigma \in I_{M/K}$ also acts trivially on $\mathcal{O}_L/\mathfrak{m}_L$ and $\sigma|_L \in I_{L/K}$.

Let $\pi_M \in M$ be a uniformiser and let r be the ramification degree of L/M. Then π_M^r is a uniformiser of L. Any element $\sigma \in P_{M/K}$ maps π_M to $\pi_M(1+m)$ for some $m \in \mathfrak{m}_M$. Then $\sigma(\pi_M) = \pi_M^r(1+m)^r$. We know that $(1+m)^r - 1 \in \mathfrak{m}_M$, hence $(1+m)^r - 1 \in \mathfrak{m}_M \cap L = \mathfrak{m}_L$. Therefore, $\sigma|_L \in P_{L/K}$. □

Let L/K be a finite Galois extension, let $\pi \in L$ be a uniformiser and let λ be the residue field of L. Let $I_{L/K}$ and $P_{L/K}$ be as in the previous definition. Then we define a morphism

$$\nu_L : I_{L/K}/P_{L/K} \longrightarrow \lambda^* : \sigma P_{L/K} \longmapsto \overline{\left(\frac{\sigma(\pi)}{\pi}\right)}.$$

Lemma 12 ([Ser, prop. 7, p. 67])**.** *The morphism ν_L is well-defined and does not depend on the choice of π. Moreover, it induces an injection*

$$I_{L/K}/P_{L/K} \to \boldsymbol{\mu}(\lambda) := \{x \in \lambda : \exists n \in \mathbb{N} : x^n = 1\}.$$

Proof. First consider another uniformiser π'. Write $\pi' = \pi \cdot u$ for some unit $u \in \mathcal{O}_L^*$. Then $\frac{\sigma(\pi')}{\pi'} = \frac{\sigma(\pi)}{\pi} \cdot \frac{\sigma(u)}{u}$. As $\sigma \in I_{L/K}$, we have $\overline{\sigma(u)} = \overline{u} \in \lambda^*$ and hence, ν_L does not depend on the choice of π.

Next, let us show that ν_L is well-defined. Suppose that $\tau \in P_{L/K}$, then $\frac{\sigma\tau(\pi)}{\pi} = \frac{\sigma\tau(\pi)}{\tau(\pi)} \cdot \frac{\tau(\pi)}{\pi}$. Now $\tau(\pi)$ is just another uniformiser, hence

$$\overline{\left(\frac{\sigma\tau(\pi)}{\tau(\pi)}\right)} = \overline{\left(\frac{\sigma(\pi)}{\pi}\right)}.$$

As $\tau \in P_{L/K}$, it acts trivially on $\mathcal{O}/\mathfrak{m}^2$ and hence $\tau(\pi) = \pi \cdot u$ for some u which is $1 \mod \mathfrak{m}$, i.e., $\overline{\left(\frac{\tau(\pi)}{\pi}\right)} = 1$.

The image of ν_L is a finite subgroup of λ^*. The injectivity is obvious: if $\sigma \in I_{L/K}$ sends π to $\pi \mod \mathfrak{m}^2$, then it acts trivially on $\mathcal{O}/\mathfrak{m}^2$. □

The following corollary is not as trivial as it looks.

Corollary 13. *The group I_K/P_K is canonically isomorphic to $\lim_{n\to\infty} \boldsymbol{\mu}_n(k^{\mathrm{sep}})$. This profinite group is isomorphic to $\hat{\mathbb{Z}}'(1) := \prod_{q \neq p} \mathbb{Z}_q(1)$, where q runs over the primes not equal to p.*

Proof. We will construct a morphism $\eta : I_K/P_K \to \lim_n \boldsymbol{\mu}_n(k^{\mathrm{sep}})$. Suppose that we have an element $(\sigma_L)_L \in I_K/P_K$. Then $\eta((\sigma_L)_L)_n \in \boldsymbol{\mu}_n(k^{\mathrm{sep}})$ will be defined as follows. Let n' be the part of n coprime to p. Take your favourite extension of K with ramification degree n', say $M := K(\sqrt[n']{\pi})$ and then let $\eta((\sigma_L)_L)_n$ be $\overline{\left(\frac{\sigma_M(\pi_M)}{\pi_M}\right)}$, where $\pi_M \in \mathcal{O}_M$ is a uniformiser. To check that this does not depend on the choice of an extension, check the compositum and use the previous lemma. The bijectivity of this map also follows from the lemma. □

Now we define $P_{K,\ell}$ to be the inverse image of $\prod_{q\neq p,\ell} \mathbb{Z}_q(1) \subset I_K/P_K$ in I_K and $G_{K,\ell} := G_K/P_{K,\ell}$.

Proposition 14. *Let V be an ℓ-adic Galois representation of G_K and let $\rho : G_K \to \mathrm{GL}(V)$ be the corresponding morphism. Then $\rho(P_{K,\ell})$ is finite.*

Proof. Let $T \subset V$ be a $\mathbb{Z}_\ell$-lattice stable under G_K and generating V. Then the image $\rho(G_K)$ is a closed subgroup of $\mathrm{Aut}_{\mathbb{Z}_\ell}(T) \cong \mathrm{GL}_h(\mathbb{Z}_\ell)$. Let $N_n \subset \mathrm{GL}_h(\mathbb{Z}_\ell)$ be the set of elements acting trivial modulo ℓ^n for $n \geqslant 1$. Now N_1/N_n is a finite group whose order is a power of ℓ and $N_1 = \lim_{n\to\infty} N_1/N_n$ is a pro-ℓ-group. Hence, it is disjoint from $\rho(P_{K,\ell})$ as $P_{K,\ell}$ is prime to ℓ. Hence, $\rho(P_{K,\ell})$ injects into $\mathrm{GL}_h(\mathbb{F}_\ell)$ under the reduction map and it is finite. □

Now let us define some more notions.

Definition 15. Let V be an ℓ-adic representation of G_K.

- We say that V is *unramified* or *has good reduction* if I_K acts trivially.
- We say that V has *potentially good reduction* if $\rho(I_K)$ is finite, i.e., if there exists a finite extension L of K in K^{sep} such that V as an ℓ-adic representation of G_L is unramified.
- We say that V is *semistable* is I_K acts unipotently, i.e., if the semisimplification of V has good reduction.
- We say that V is *potentially semistable* if there exists a finite extension L of K in K^{sep} such that V is semistable as a representation of G_L, or, equivalently, if the semisimplification of G_K has potentially good reduction.

The following theorem is needed to prove the main result of this lecture.

Theorem 16 ([FoOu, th. 1.24])**.** *Assume that the group*

$$\boldsymbol{\mu}_{\ell^\infty}(K(\boldsymbol{\mu}_\ell)) = \{x \in K(\boldsymbol{\mu}_\ell) : \exists n \in \mathbb{N} : x^{\ell^n} = 1\}$$

is finite. Then any ℓ-adic representation of G_K is potentially semistable. As $\boldsymbol{\mu}_{\ell^\infty}(k) \cong \boldsymbol{\mu}_{\ell^\infty}(K)$, this is the case if k is finite.

Proof. Let V be any ℓ-adic representation of G_K. By proposition 14 we know that $P_{K,\ell}$ has finite image. Hence, by extending K if necessary, we may and will assume that $P_{K,\ell}$ acts trivially. Now the action of G_K on V factors through $G_{K,\ell}$. We have an exact sequence

$$1 \longrightarrow \mathbb{Z}_\ell(1) \longrightarrow G_{K,\ell} \longrightarrow G_k \longrightarrow 1.$$

Now let $t \in \mathbb{Z}_\ell(1)$ be a topological generator and consider its action on V. We choose a finite extension E of $\mathbb{Q}_\ell$ such that the characteristic polynomial of t splits in linear factors. Now let $V' = E \otimes_{\mathbb{Q}_\ell} V$ and consider it as $G_{K,\ell}$-module.

Let $a \in E$ be an eigenvalue of t and let $v \in V' \setminus 0$ be an eigenvector. As $\mathbb{Z}_\ell(1)$ is normal, $gtg^{-1} \in \mathbb{Z}_\ell(1)$ and is of the form $t^{\chi_\ell(g)}$ for some character function $\chi_\ell : G_{K,\ell} \to \mathbb{Z}_\ell^*$. Now we have

$$t \cdot g^{-1}(v) = g^{-1}(gtg^{-1})v = a^{\chi_\ell(g)} \cdot g^{-1}v.$$

Hence, $a^{\chi_\ell(g)}$ is also an eigenvalue of t. Now the goal is to prove that $\chi_\ell(g)$ can take infinitely many values and to prove that a is a root of unity.[*1]

1: In the text they suggest that $\mathrm{Im}(\chi_\ell)$ is open as $\boldsymbol{\mu}_{\ell^\infty}(K(\boldsymbol{\mu}_\ell))$ is finite. I don't know why that would be true. If somebody knows, please let me know.

We have the following inclusion of groups and corresponding diagram of fields:

$$1 \subset P_K \subset P_{K,\ell} \subset I_K \subset G_K;$$

$$K^{\mathrm{sep}} \supset K^{\mathrm{wild}} \supset K^{\mathrm{wild},\ell} \supset K^{\mathrm{unr}} \supset K.$$

As $p \neq \ell$, the ℓ^n-th roots of unity are all in K^{unr}. Consider the extension $L = K[X]/(X^{\ell^n} - \pi) \subset K^{\mathrm{wild},\ell}$ for $n \in \mathbb{N}$ and let $\alpha := \overline{X} \in L$. Then $g^{-1}\alpha = \zeta_{\ell^n} \cdot \alpha$ for some ℓ^n-th root of unity $\zeta_{\ell^n} \in K^{\mathrm{sep}}$. As $t \in I_K/P_{K,\ell}$ it acts trivially on ζ_{ℓ^n} and $t\alpha = \zeta_{\ell^n}^{t_n}\alpha$ for some $(t_n) \in \mathbb{Z}_\ell^*$. Hence,

$$gtg^{-1}\alpha = g(\zeta_{\ell^n}^{t_n+1}\alpha) = g(\zeta_{\ell^n})^{t_n}\alpha.$$

The condition in the statement, $|\boldsymbol{\mu}_{\ell^\infty}(K(\boldsymbol{\mu}_\ell))| < \infty$, implies that $K(\boldsymbol{\mu}_{\ell^\infty})/K$ is an infinite extension. Hence, gtg^{-1} takes infinitely many values when we let g range over $G_{K,\ell}$. In particular, $\chi_\ell(g)$ takes infinitely many values. Hence, a is a root of unity and some power t^N of t acts unipotently. Hence, some subgroup of finite index of $\mathbb{Z}_\ell(1)$ acts unipotently. In particular, some subgroup of finite index of I_K acts unipotently and the statement follows. □

Now we are ready to state and prove the main theorem.

Theorem 17 (Grothendieck's ℓ-adic monodromy)**.** *Let K be a local field. Then all ℓ-adic representations of G_K that we have seen in the previous section ($V_\ell(A)$, $H^m_{\acute{e}t}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell), \ldots$) are potentially semistable.*

Proof. Let X be projective smooth over K. Then X can be described by finitely many equations and is defined over some field K_0 of finite type over the prime field of K. Let K_1 be the (topological) closure of K_0 in K. It is a complete discrete valuation field with residue field k_1 of finite type over $\mathbb{F}_p$.

Now let $k_2 = k_1^{p^{-\infty}}$ be the perfect closure[*2] of k_1. Then construct a complete discrete valuation field K_2 in K containing K_1 with residue field k_2. Then $\boldsymbol{\mu}_{\ell^\infty}(k_2) = \boldsymbol{\mu}_{\ell^\infty}(k_1)$ is finite. Then the action of G_K on V comes from the action of G_{K_2} and we can use the previous theorem. □

2: This was wrong in [FoOu]: he took the radical closure. See also [Esn].

Also the contrary is sometimes true.

Theorem 18 ([FoOu, th. 1.26])**.** *Assume k is algebraically closed. Then any potentially semistable ℓ-adic representation of G_K is one of the mentioned ones.*

Proof. Omitted. □

References

[Esn] H. Esnault, *Forschungsseminar über Fontaine-Ouyang*: ℓ-adic Galois representations, 2008.

[FoOu] J.-M. Fontaine, Y. Ouyang, *Theory of p-adic Galois Representations*. Springer.

[Mil] J.S. Milne, *Abelian Varieties*, version 2.0. <http://www.jmilne.org/math/CourseNotes/av.html>.

[Mum] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research, 1970.

[Ser] J.P. Serre, *Local fields*, Springer-Verlag, 1980.